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# Two-level atom in a cross cavity

## Átomo de dos niveles en una cavidad en cruz

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## ABSTRACT

In this work we propose a model to analyze the interaction of a two-level atom (TLA) with two electromagnetic fields. The interaction occurs within a cavity with a cross configuration. The wavefunction of the system is calculated via time-evolution operator. An interaction Hamiltonian that reassembles the well-known Jaynes-Cummings model is obtained by applying a canonical transformation to the field operators.



## RESUMEN

En este trabajo proponemos un modelo para analizar la interacción de un átomo de dos niveles (TLA, por sus siglas en inglés) con dos campos electromagnéticos. La interacción ocurre dentro de una cavidad en una configuración en forma de cruz. La función de onda del sistema es calculada mediante el operador de evolución. Un Hamiltoniano de interacción que proporciona el bien conocido modelo de Jaynes-Cummings es obtenido mediante la aplicación de una transformación canónica a los operadores del campo.


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**Palabras clave:**
Átomo de dos niveles; modelo de Jaynes- Cummings; cavidad en forma de cruz.

## INTRODUCTION

The Jaynes-Cummings model (JCM) is the most important and the simplest model for explaining the matter-radiation interaction which can be solved analytically (Shore & Knight, 1993). Generalizations of this model have been developed by letting the two level atom (TLA) (figure 1) interact with a two-mode field (Abdalla, Abdel-Aty & Obada, 2002; Marchiolli, Missori & Roversi, 2003). The proposal is an extension of the JCM consisting of a TLA inside a cross cavity configuration.

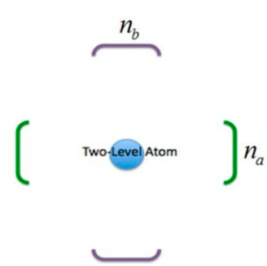


**Figure 1.** This is an schematic representation of the physical problem.
Source: Authors own elaboration.

An electric field, which field operators will be labeled as $a$ and $a^{\dagger}$, is injected in the $x$ direction while other electric field, which operators are labeled as $b$ and $b^{\dagger}$, is injected in the $y$ direction. The corresponding atomic dipole has quantum components in the direction $x$ and $y$ given by the usual Pauli's matrixes $\sigma_x$ and $\sigma_y$ while the atomic inversion is given by the Pauli's matrix $\sigma_z$.

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## Mathematical model

In the dipole approximation the Hamiltonian that describes the system is given by the following equation:

$$H = H_0 + \hbar g_x \sigma_x \left(a + a^\dagger\right) + \hbar g_y \sigma_y \left(b + b^\dagger\right). \tag{1}$$

Here the free Hamiltonian of the system is given by $H_0 = \hbar\omega_0\sigma_z/2 + \hbar\omega a^\dagger a + \hbar\omega b^\dagger b$ where $\omega_0$ is the atomic transition frequency and $\omega$ is the frequency for each electric field. The coupling constants between the fields and the electric dipole in the $x$ and $y$ direction are $g_x$ and $g_y$ respectively. Expressing equation (1) in terms of the raising and lowering operators $\sigma_+$ and $\sigma_-$ and by using the rotating wave approximation (RWA) we obtain the following expression for the Hamiltonian:

$$\begin{aligned} H_0 &= \hbar\omega_0\sigma_z/2 + \hbar\omega a^\dagger a + \hbar\omega b^\dagger b \\ &+ \hbar g_x \left(\sigma_+ a + a^\dagger \sigma_-\right) \\ &- i\hbar g_y \left(\sigma_+ b + b^\dagger \sigma_-\right). \end{aligned} \tag{2}$$

To find the time evolution of the system by using the Hamiltonian is complicated, we will introduce a representation in which the Hamiltonian acquires the form of the single-mode Jaynes-Cummings model. Such new representation will be defined by the rotation

$$\begin{pmatrix} A \\ iB \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} a \\ ib \end{pmatrix}. \tag{3}$$

An effective coupling constant have been defined as $g_{\text{eff}} = \sqrt{g_x^2 + g_y^2}$. The angles are related with the coupling by $\cos\phi = g_x / g_{\text{eff}}$ and $\sin\phi = g_y / g_{\text{eff}}$. The new operators $A$ and $B$ obey the same commutation rules that $a$ and $b$, *i.e.* $[A, A^\dagger] = 1$, $[B, B^\dagger] = 1$ and $[A, B] = 0$. Also the photon number operator remains unchanged $a^\dagger a + b^\dagger b = A^\dagger A + B^\dagger B$. The obtained effective Jaynes-Cummings Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega A^\dagger A + \hbar\omega B^\dagger B \\ &+ \frac{1}{2}\hbar\Delta\sigma_z + \hbar g_{\text{eff}}\left(A\sigma_+ + A^\dagger\sigma_-\right). \end{aligned} \tag{4}$$

Here the detuning is $\Delta = \omega_0 - \omega$.

## Time evolution operator

With the definitions of the operators $A$ and $B$ the initial Hamiltonian acquires the form of the well known Jaynes–Cummings model. Due to this we identify the operators $N_A = A^\dagger A + \sigma_+\sigma_-$, $C_A = 1/2\,\hbar\Delta\sigma_z + \hbar g_{\text{eff}}\left(A\sigma_+ + A^\dagger\sigma_-\right)$ and define the additional term as $n_B = B^\dagger B$. In this way we write the Hamiltonian using constant operators

$$H = \hbar\omega n_B + \hbar\omega N_A + C_A. \tag{5}$$

To write the Hamiltonian by using constant operators ($[H,n_B] = [H,N_A] = [H,C_A] = 0$) allows us to readily calculate the time evolution operator which is given by the following equation:

$$U(t) = e^{-in_B t} e^{-iN_A t} e^{-iC_A t}. \tag{6}$$

The first two terms do not contribute to the dynamics of the system in this representation. The term that contributes significantly to the dynamics of the system is the exponential $e^{-iC_A t}$ which can be calculated by Taylor's series

$$\begin{aligned} e^{-iC_A t} &= \cos\left(t g_{\text{eff}}\sqrt{N_A}\right) \\ &- i\frac{A^\dagger}{\sqrt{N_A+1}}\sin\left(t g_{\text{eff}}\sqrt{N_A+1}\right)\sigma_- \\ &- \frac{i}{\sqrt{N_A}}\sin\left(t g_{\text{eff}}\sqrt{N_A}\right)A\sigma_+. \end{aligned} \tag{7}$$

Exact resonance has been considered.

## DISCUSSION

We have simplified the problem by applying a rotation to the field operators. Now it is necessary to determinate how related are the states in the original representation with the states in the new representation.

### Fock states

The requirement here is to determinate eigenstates $|m\rangle$ and $|n\rangle$ of the number operators $a^\dagger a$ and $b^\dagger b$ with the eigenstates $|m\rangle\rangle$ and $|n\rangle\rangle$ of the new number operators $A^\dagger A$ and $B^\dagger B$. The fock states can be obtained from the vacuum state by applying the $a^\dagger$ and $b^\dagger$ operators:

$$\left| m, n \right\rangle = \frac{\left(a^\dagger\right)^m \left(b^\dagger\right)^n}{\sqrt{m!}\sqrt{n!}}\left| 0, 0 \right\rangle. \tag{8}$$

Here we can recognize the equality between the vacuum state of the operators $n_a$ and $n_b$ denoted as $|0,0\rangle$ and the vacuum state of the operators $n_A$ and $n_B$ denoted as $|0,0\rangle\rangle$, *i.e.*, $|0,0\rangle = |0,0\rangle\rangle$. And by using the equation (3) we realize that the Fock state in the new representation becomes a state with binomial distribution.


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$$\left| m, n \right\rangle \to \sum_{k=0}^{m} \sum_{l=0}^{n} \frac{\sqrt{m!n!}\sqrt{\left(m+n-k-l\right)!\left(k+l\right)!}}{k!l!\left(m-k\right)!\left(n-l\right)!} \times \left(-i\right)^{k+n-l} \left(\sin\phi\right)^{n+k-l} \left(\cos\phi\right)^{m-k+l} \times \left| m+n-k-l, k+l \right\rangle\!\Big\rangle. \tag{9}$$

**Coherent states**

To determinate how related are a pair of coherent states in the original representation $\left|\alpha, \beta\right\rangle$ with the states in the new representation consider the coherent states written as the displacement operator acting on the vacuum state

$$\left|\alpha, \beta\right\rangle = D\left(\alpha\right) D\left(\beta\right) \left|0, 0\right\rangle. \tag{10}$$

Where the displacement operators are given by $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ and $D(\beta) = \exp(\beta b^\dagger - \beta^* b)$. Again using equation (3) we find that the state $\left|\alpha, \beta\right\rangle$ in the new representation also is a coherent state

$$\left|\alpha, \beta\right\rangle \to \left|\alpha_A, \beta_B\right\rangle\!\Big\rangle, \tag{11}$$

where $\alpha_A = \alpha \cos\phi - i\beta \sin\phi$ and $\beta_B = \beta\cos\phi - i\alpha\sin\phi$.

Note that the results obtained are closely related with the photon distribution obtained in experiments with light passing through a beam splitter (Gerry & Knight, 2005).

## RESULTS

With knowledge of the transformation rule for a pair of Fock states we can determine the transformation rule for more complex states of the electric field such as $\left|1001\right\rangle = \left(\left|1, 0\right\rangle + \left|0, 1\right\rangle\right)/\sqrt{2}$. In this case the state in the new representation acquires a phase which in the calculation of the atomic inversion is irrelevant (figure 2). For an atom initially excited and with the field in the state $\left|1001\right\rangle$ the state in the new representation is given by

$$\left|\psi(t=0)\right\rangle = e^{-i\phi} \frac{\left|1, 0, e\right\rangle + \left|0, 1, e\right\rangle}{\sqrt{2}}. \tag{12}$$

On the other hand the equation 9 allows to see that for a field state $\left|m, 0, e\right\rangle$ the corresponding state in the new representation will have a binomial distribution

$$\left|\psi_{\text{field}}\left(t=0\right)\right\rangle\!\Big\rangle = \sum_{k=0}^{m} \sqrt{\frac{m!}{k!\left(m-k\right)!}} \left(-i\right)^k \left(\sin\phi\right)^k \times \left(\cos\phi\right)^{m-k} \left|m-k, k, e\right\rangle\!\Big\rangle. \tag{13}$$

Again the TLA has been assumed initially excited.

## CONCLUSIONS

The introduction of the operators $A$ and $B$ allowed us to write the Hamiltonian as an effective single mode Jaynes-Cummings system. Such Hamiltonian was

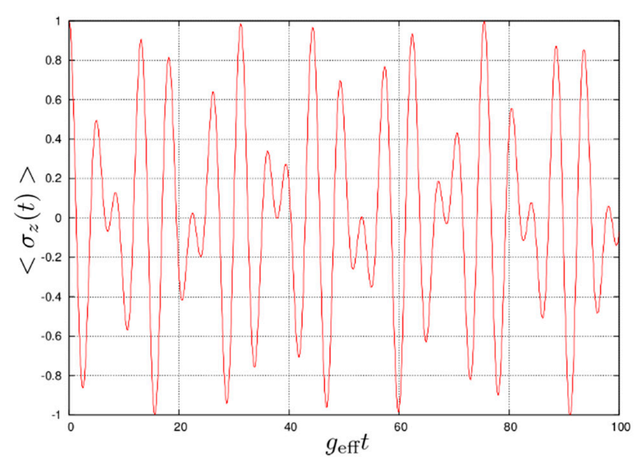


**Figure 2.** Atomic inversion $\left\langle\psi(t)\right| \sigma_z \left|\psi(t)\right\rangle$ when the atom is prepared in the excited state and when the field is an entangled single photon state $\left(\left|1, 0, e\right\rangle + \left|1, 0, e\right\rangle\right)/\sqrt{2}$.
Source: Authors own elaboration.

written in terms of constant operators that enabled the calculation of the propagator. The new representation introduced gives photon distribution in a similar fashion to the photon distribution obtained with beam splitters.

In this representation we calculated the wave function, for a state in the weak field regime, by applying the time evolution operator.

The atomic inversion for the initial field in a state on the form $\left|m,0\right\rangle$ collapses and revivals because of the binomial distribution of the modes of the field (figure 3).

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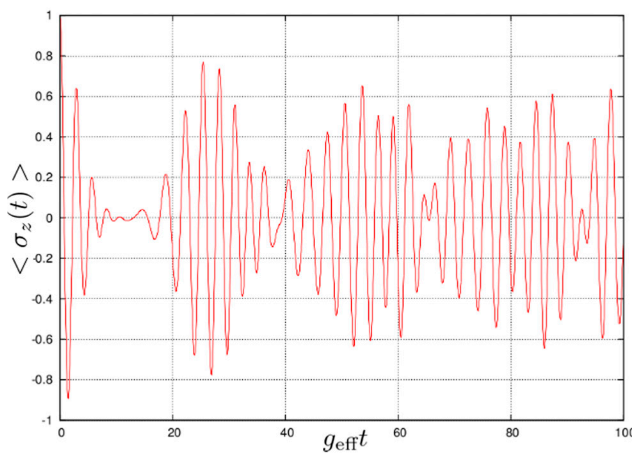


**Figure 3.** Atomic inversion when the atom is prepared in the excited state and when the field is an entangled photon state $\left|7,0,e\right\rangle$.
Source: Authors own elaboration.