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Simplified Method of Characteristics for Fast Transient Solution of Frequency and Space Dependent Transmission Lines

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1. Abstract

In this work, the solution of the non uniform line with frequency dependence via a simplified method of characteristics is described. Using this method, a line with symmetric non uniformity with respect to its center can be solved identically to a uniform line, obtaining for that purpose transient parameters of series resistance and transversal conductance, which define frequency and space dependence of the line. Besides, space discretization required by the conventional method of characteristics is replaced by a simple division of the line in two equal segments. Accuracy of the method is validated with 3 application examples, comparing the results from those obtained by the conventional method of characteristics, the Numerical Laplace Transform and an experimental measurement published elsewhere.

Key words: frequency dependent parameters, method of characteristics, non uniform line.

2. Resumen (Método de características simplificado para la solución transitoria rápida de líneas de transmisión dependientes de la frecuencia y el espacio)

En este trabajo se describe la solución de la línea no uniforme con dependencia frecuencial por medio de un método de características simplificado. Mediante este método, es posible resolver una línea con no uniformidad simétrica respecto a su centro de manera idéntica a una línea uniforme, obteniendo para ello los parámetros transitorios de resistencia serie y conductancia transversal que definen la dependencia tanto frecuencial como espacial de la línea. Además, la discretización en la distancia utilizada por el método de características convencional se sustituye por una simple división de la línea en dos segmentos iguales. La precisión del método se valida mediante tres ejemplos de aplicación, comparando los resultados con los obtenidos por el método de características convencional, el algoritmo de la transformada numérica de Laplace y una medición experimental publicada previamente.

Palabras clave: parámetros dependientes de la frecuencia, método de características, línea no uniforme.

3. Introduction

Transmission line modeling for electromagnetic transient studies is usually performed neglecting variations of the line electrical parameters with distance, this is, the line is assumed with constant cross-section and constant electrical properties of conductors and dielectric. However, for fast transient studies, such as those due to lightning or substation grounding, the correct description of these non uniformities can be as important as the inclusion of frequency dependence of the line parameters (mainly due to skin effect in conductors and ground return), yielding significant differences of the resulting wave shapes.

The Non Uniform Line (NUL) problem has been analyzed in the last decades using frequency domain as well as time domain techniques [1-10]. The advantage of frequency domain techniques is that the line frequency dependence can be taken into account in a straightforward manner; nevertheless, dealing with changes in the network topology can be complicated [11]. Conversely, time domain methods, such as EMTP-type programs or finite difference algorithms, require several approximations in order to take into account the frequency dependence of the line. It has been reported that even the most advanced line models available in commercial programs can be prone to errors [11].

Modeling of NULs using EMTP-type programs is not an easy task. A division of the line in uniform line segments is required, considering the parameters variation between them. This yields a large and highly dispersive admittance matrix; therefore, both the modeling and simulation procedures are in general cumbersome and time-consuming, and a great deal of experience is needed in order to define the optimal number of line sections.

In the other hand, from the existing methods based on finite difference algorithms, the method of characteristics is usually preferred given its proven efficiency to solve hyperbolic type problems, such as the well known Telegrapher equations that describe wave propagation along transmission lines. This method has been successfully applied for the solution of transmission lines with corona and frequency dependence [12-13], non uniformities [4-5, 14], as well as transmission tower modeling [10] and field excited lines [15].

An important problem related to the NUL solution with the method of characteristics is that the discretization grid, also known as characteristics grid, becomes irregular, taking into account that the characteristics that define this grid, being functions of x, become curves. This problem has been solved in previous works establishing a regular grid of points in the x-t plane and then forcing the characteristics to cross these points [4-5, 14]. However, determination of these points involves an iterative procedure prior to the simulation.

In this work, to major simplifications to the method of characteristics for the solution of symmetrical NULs with frequency dependence are proposed:

1. Elimination of the interpolation procedure by means of synthesizing a uniform line model equivalent to the NUL model.
2. Elimination of the line discretization, defining a new grid for which only a division in two segments is required.

4. Development
4.1 Conventional method of characteristics

In the following sections, conventional solutions for frequency dependent line (uniform and non-uniform) are described.

4.1.1. Frequency dependent uniform line

The modified Telegrapher equations for a frequency dependent uniform line were defined by Radulet *et al.* [16] as follows

$$\frac{\partial v(x,t)}{\partial x} + L_G \frac{\partial i(x,t)}{\partial t} + \frac{\partial}{\partial t} r'(t) * i(x,t) = 0 \quad (1a)$$

$$\frac{\partial i(x,t)}{\partial x} + C_G \frac{\partial v(x,t)}{\partial t} + \frac{\partial}{\partial t} g'(t) * v(x,t) = 0 \quad (1b)$$

where $v(x,t)$ and $i(x,t)$ are the voltages and currents along the line; L_G and C_G are the geometric values of inductance and capacitance, while $r'(t)$ and $g'(t)$ represent transient values of resistance and conductance of the line. In frequency domain, (1a) and (1b) are given by

$$\frac{dV(x,s)}{dx} + s[L_G + R'(s)]I(x,s) = \frac{dV(x,s)}{dx} + Z(s)I(x,s) = 0 \quad (2a)$$

$$\frac{dI(x,s)}{dx} + s[C_G + G'(s)]V(x,s) = \frac{dI(x,s)}{dx} + Y(s)V(x,s) = 0 \quad (2b)$$

being $Z(s)$ and $Y(s)$ the series impedance and shunt conductance of the line, per unit length. Transient parameters $R'(s)$ and $G'(s)$ can be rationally fitted as

$$R'(s) = \frac{k_0}{s} + \sum_{i=1}^{N} \frac{k_i}{s + p_i} + k_\infty \quad (3a)$$

$$G'(s) = \frac{m_0}{s} + \sum_{i=1}^{N} \frac{m_i}{s + q_i} + m_\infty \quad (3b)$$

Pairs $\{k_i, p_i\}$ and $\{m_i, q_i\}$ represent the i-th poles and residues of the respective partial fraction expansion. If the Leibnitz rule [4] is applied, (1a) and (1b) can be rewritten as follows

$$\frac{\partial v(x,t)}{\partial x} + L_{cor} \frac{\partial i(x,t)}{\partial t} + R_x i(x,t) + \psi = 0 \quad (4a)$$

$$\frac{\partial i(x,t)}{\partial x} + C_{cor} \frac{\partial v(x,t)}{\partial t} + G_x i(x,t) + \phi = 0 \quad (4b)$$

where:

$$L_{cor} = k + L_G, \quad C_{cor} = m + C_G \qquad (5a),(5b)$$

$$R_x = \sum_{i=0}^{N_1} k_i, \quad G_x = \sum_{i=0}^{N_2} m_i \qquad (6a),(6b)$$

Expressions Ψ and ϕ represent convolution terms, which are computed from the rational approximations of $R'(s)$ and $G'(s)$ using a recursive algorithm [17], as described in the Appendix. Rewriting (4a) and (4b) in compact form:

$$\frac{\partial \mathbf{U}}{\partial x} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{BU} + \mathbf{W} = \mathbf{0} \qquad (7)$$

where

$$\mathbf{U} = \begin{bmatrix} v \\ i \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & L_{cor} \\ C_{cor} & 0 \end{bmatrix} \qquad (8a),(8b)$$

$$\mathbf{B} = \begin{bmatrix} 0 & R_x \\ G_x & 0 \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} \psi \\ \phi \end{bmatrix} \qquad (8c),(8d)$$

Eigenvalues ($\lambda_{1,2}$) and eigenvectors ($\mathbf{M}_L$ and $\mathbf{M}_R$) of $\mathbf{A}$ are given by

$$\lambda_{1,2} = \pm\sqrt{L_{cor}C_{cor}} \qquad (9)$$

$$\mathbf{M}_L = \begin{bmatrix} 1 & Z_W \\ 1 & -Z_W \end{bmatrix}, \quad \mathbf{M}_R = \begin{bmatrix} 1 & 1 \\ Y_W & -Y_W \end{bmatrix} \qquad (10a),(10b)$$

where

$$Z_W = \sqrt{L_{cor}/C_{cor}}, \quad Y_W = Z_W^{-1} \qquad (11a),(11b)$$

Left multiplying (4a,b) times $\mathbf{M}_L$, regrouping and applying (9) and (11) yields

$$\left[\frac{\partial}{\partial x} + \lambda_1 \frac{\partial}{\partial t}\right] v + Z_W \left[\frac{\partial}{\partial x} + \lambda_1 \frac{\partial}{\partial t}\right] i + R_x i + Z_W G_x v + \psi + Z_W \phi = 0$$

$$\left[\frac{\partial}{\partial x} + \lambda_2 \frac{\partial}{\partial t}\right] v + Z_W \left[\frac{\partial}{\partial x} + \lambda_2 \frac{\partial}{\partial t}\right] i + R_x i + Z_W G_x v + \psi + Z_W \phi = 0$$

(12a),(12b)

Describing a new coordinate system along the lines defined by $\lambda = \pm dt/dx$, called *characteristics*, the following equivalence can be applied:

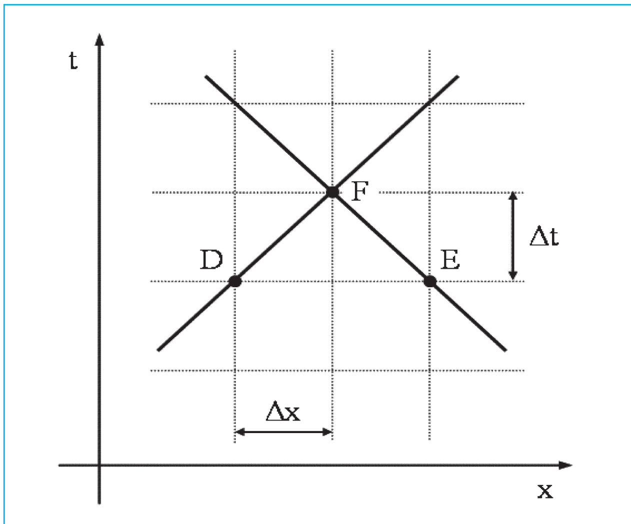

Fig. 1. Characteristics grid - Internal points.

$$\left[\frac{\partial}{\partial x} \pm \lambda_{1,2} \frac{\partial}{\partial t}\right] = \pm\frac{d}{dx} \qquad (13)$$

Using (13) in (12a,b) and expressing the derivatives in discrete form:

$$\Delta v + Z_W \Delta i + R_x i \Delta x + Z_W G_x v \Delta x + (\psi + Z_W \phi)\Delta x = 0 \qquad (14a)$$

$$\Delta v - Z_W \Delta i + R_x i \Delta x - Z_W G_x v \Delta x + (\psi - Z_W \phi)\Delta x = 0 \qquad (14b)$$

Applying finite differences to (14a,b) according to Fig. 1, values at point F (corresponding to internal points at time t) are computed using known values at points D and E (corresponding to previous time step $t-\Delta t$):

$$\alpha_1 v_F + Z_1 i_F - \alpha_2 v_D - Z_2 i_D + \frac{\Delta x}{2}(\psi_F + \psi_D) + \frac{Z_W \Delta x}{2}(\phi_F + \phi_D) = 0$$

$$\alpha_1 v_F + Z_1 i_F - \alpha_2 v_E + Z_2 i_E - \frac{\Delta x}{2}(\psi_F + \psi_E) + \frac{Z_W \Delta x}{2}(\phi_F + \phi_E) = 0$$

(15a),(15b)

where

$$\alpha_1 = 1 + \frac{Z_W G_X \Delta x}{2}, \quad \alpha_2 = 1 - \frac{Z_W G_X \Delta x}{2} \qquad (16a),(16b)$$

$$Z_1 = Z_W + \frac{R_X \Delta x}{2}, \quad Z_2 = Z_W - \frac{R_X \Delta x}{2} \qquad (16c),(16d)$$

Voltage at point F is obtained adding (15a) and (15b):

$$v_F = \frac{1}{2\alpha_1}[\alpha_2(v_D+v_E)+Z_2(i_D+i_E) - \frac{\Delta x}{2}(\psi_D+\psi_E) - \frac{Z_W\Delta x}{2}(2\phi_F+\phi_D+\phi_E)] \quad (17)$$

while the corresponding current is obtained subtracting the same equations:

$$i_F = \frac{1}{2Z_1}[\alpha_2(v_D+v_E)+Z_2(i_D+i_E) - \frac{\Delta x}{2}(2\psi_F+\psi_D+\psi_E) - \frac{Z_W\Delta x}{2} \quad (18)$$

Solution for boundary points is computed according to Fig. 2. For the sending point S (x=0), the application of an ideal voltage source $v_S = f(t)$ is considered. Applying (15b), replacing subscript F by S and solving for i_S:

$$i_S = \frac{1}{Z_1}[\alpha_2 v_S - \alpha_2 v_E + Z_2 i_E - \frac{\Delta x}{2}(\psi_S+\psi_E) + \frac{Z_W\Delta x}{2}(\phi_S+\phi_E)] \quad (19)$$

For the load boundary point L ($x = L$), the connection of a resistive load R_L such that $i_L = v_L/R_L$, is considered. Eq. (15a) is applied, replacing subscript F by L and solving for v_L:

$$v_L = \frac{R_L}{\alpha_1 R_L + Z_1}[\alpha_2 v_D - \alpha_2 V_E - \frac{\Delta x}{2}(\psi_L+\psi_D) - \frac{Z_W\Delta x}{2}(\phi_L+\phi_D)] \quad (20)$$

In order to assure convergence of the solution, discretization of x and t must comply with Courant-Friedrichs-Lewy condition (CFL) [18]: $\Delta x/\Delta t \leq \upsilon$, being υ the propagation velocity of the line.

4.1.2 Frequency dependent non uniform line

Electrical parameters of a non uniform line (NUL) are not only frequency but also distance varying. Therefore, values of λ_1 and λ_2 are functions of x, causing characteristics to be curved and generating an irregular grid, as shown in Fig. 3. This has been previously overcome by establishing a new grid of regular points and forcing characteristics to cross these points [14]. However, to determine location of crossings an iterative interpolation procedure is required before starting the transient simulation. This, according to the line non uniformity, can be computationally expensive.

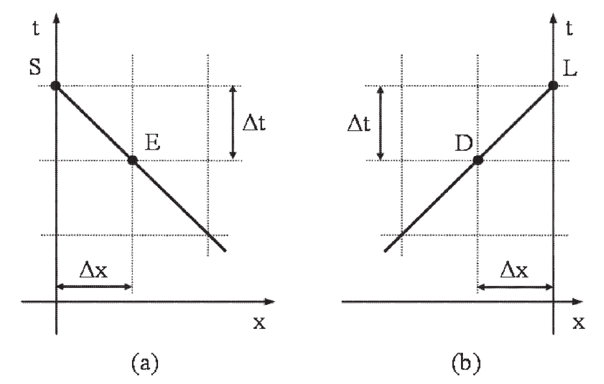

Fig. 2. Characteristics grid-Boundary points.

4.2 Simplified method of characteristics

For a line with symmetrical non uniformity with respect to its center (hereafter defined as symmetrical NUL, *e.g.*, sagging between towers), interpolation process can be avoided synthesizing a uniform line model from the chain matrix of a NUL. Moreover, for this type of line it is possible to eliminate the discretization in x by dividing the line only in two segments. By avoiding interpolation and discretization processes, line solution becomes faster and simpler.

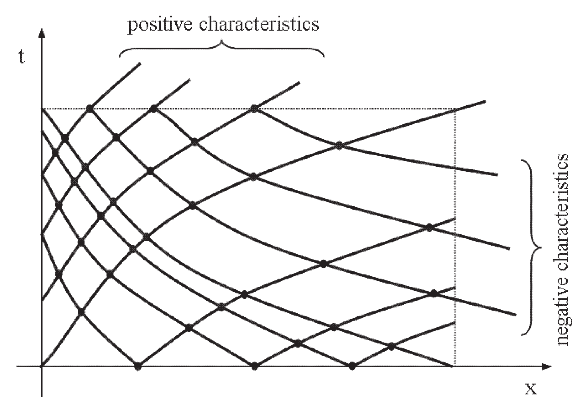

Fig. 3. Irregular grid for non uniform lines.

4.2.1 Elimination of interpolation process

According to [2-3, 19], a non uniform line can be approximated by cascaded connection of chain matrices of uniform line segments, considering the variation of the line electrical parameters between each subdivision. Thus, the NUL can be defined in the frequency domain as

$$\begin{bmatrix} V(L,s) \\ I(L,s) \end{bmatrix} = \mathbf{\Phi}^{(M)} \ldots \mathbf{\Phi}^{(i)} \ldots \mathbf{\Phi}^{(1)} \begin{bmatrix} V(0,s) \\ I(0,s) \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} V(0,s) \\ I(0,s) \end{bmatrix} \tag{21}$$

where $V(0,s)$ and $V(L,s)$ are the voltages at the line ends (x=0 and x=L, being L the line length); $I(0,s)$ and $I(L,s)$ represent currents at the same points; $\mathbf{\Phi}^{(i)}$ is the chain matrix of the i-th line segment, while Φ_{11}, Φ_{12}, Φ_{21} and Φ_{22} are the elements of the chain matrix of the complete NUL, defined by

$$\begin{bmatrix} \Phi_{11} & \Phi_{11} \\ \Phi_{11} & \Phi_{11} \end{bmatrix} = \begin{bmatrix} \cosh(\gamma_{nu} L) & Y_{0,nu}^{-1} \operatorname{senh}(\gamma_{nu} L) \\ Y_{0,nu} \operatorname{senh}(\gamma_{nu} L) & \cosh(\gamma_{nu} L) \end{bmatrix} \tag{22}$$

being γ_{nu} the propagation constant and $Y_{0,nu}$ the characteristic admittance of the complete NUL. These values can be computed from (22):

$$Y_{0,\,nu} = \sqrt{\Phi_{21} / \Phi_{12}} \tag{23a}$$

$$\gamma_{nu} = \operatorname{arccosh}(\Phi_{11})/L = \operatorname{arccosh}(\Phi_{22})/L \tag{23b}$$

Previous variables can also be defined from the series impedance and shunt admittance of the NUL, Z_{nu} and Y_{nu}:

$$Y_{0,\,nu} = \sqrt{Y_{nu} / Z_{nu}} \tag{24a}$$

$$\gamma_{nu} = \sqrt{Z_{nu}\, Y_{nu}} \tag{24b}$$

Solving (24a,b) for Z_{nu} and Y_{nu} and applying (23a,b):

$$Z_{nu} = \sqrt{\frac{\Phi_{12}}{\Phi_{21}}}\; \frac{\operatorname{arccosh}(\mathbf{\Phi}_{11})}{L} \tag{25a}$$

$$Y_{nu} = \sqrt{\frac{\Phi_{21}}{\Phi_{12}}}\; \frac{\operatorname{arccosh}(\mathbf{\Phi}_{11})}{L} \tag{25b}$$

According to (25a,b), a uniform line model can be completely defined from the chain matrix of a NUL. This model can be described similarly to (2a,b) replacing Z and Y by Z_{nu} and Y_{nu}. In the other hand, transient parameters of the NUL are computed as

$$R'(s) = Z_{nu} / s \tag{26a}$$

$$G'(s) = Y_{nu} / s \tag{26b}$$

were L_G and C_G are obtained for a mean value of the line non uniformity. Starting from the rational approximation of $R'(s)$ and $G'(s)$ of (26a,b), solution via the method of characteristics is identical from that of a uniform line, *i.e.*, the characteristics are straight and produce a regular grid, so that the interpolation described in section 4.1.2 is not required.

4.2.2 Elimination of discretization process

In order to avoid the discretization in x required by the conventional method of characteristics, the grid shown in Fig. 4 is used, wherein the line is divided in two equal segments, solving only for 3 points at each time step: the boundaries and one internal point. Solution for these points at time t is computed from values known at $t-\tau/2$, being τ the travel time of the line, so the Courant-Friedrichs-Lewy condition is still satisfied. For the boundary points this yields

$$i_S = \frac{1}{Z_1}\Big[\alpha_1 v_S - \alpha_2 v_{F-\tau/2} + Z_2 i_{F-\tau/2} - \frac{L}{4}(\psi_S + \psi_{F-\tau/2}) + \frac{Z_W L}{4}(\phi_S + \phi_{F-\tau/2})\Big] \tag{27a}$$

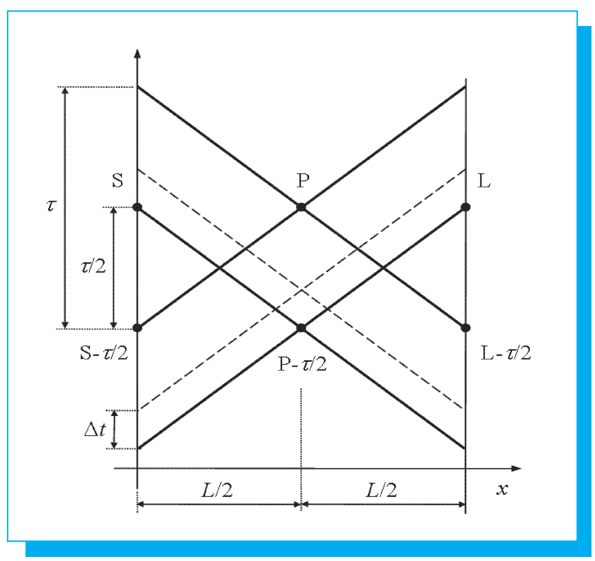

Fig. 4. Simplified characteristics grid.

$$v_L = \frac{R_L}{\alpha_1 R_L + Z_1}[\alpha_2 v_{F-\tau/2} + Z_2 i_{F-\tau/2} - \frac{L}{4}(\psi_L + \psi_{F-\tau/2}) - \frac{Z_W L}{4}(\phi_L + \phi_{F-\tau/2})] \quad (27b)$$

and for the internal point

$$i_S = \frac{1}{2\alpha_1}[\alpha_2(v_{S-\tau/2} - v_{L-\tau/2}) + Z_2(i_{S-\tau/2} - i_{L-\tau/2}) - \frac{L}{4}(\psi_{S-\tau/2} + \psi_{L-\tau/2}) - \frac{Z_W L}{4}(\phi_F + \phi_{S-\tau/2} + \phi_{L-\tau/2})] \quad (28a)$$

$$i_S = \frac{1}{2Z_1}[\alpha_2(v_{S-\tau/2} - v_{L-\tau/2}) + Z_2(i_{S-\tau/2} + i_{L-\tau/2}) - \frac{L}{4}(2\psi_F + \psi_{S-\tau/2} + \psi_{L-\tau/2}) - \frac{Z_W L}{4}(\phi_{S-\tau/2} - \phi_{L-\tau/2})] \quad (28b)$$

Equations (27a,b) and (28a,b) are solved at each time step Δt similarly to the conventional method of characteristics.

4.3 Validation of the proposed method

Accuracy of the proposed method is validated by means of 3 application examples:

" Sagging between 2 towers
" Simulation of a field experiment
" Step response of a machine winding

Results are compared with those obtained by the conventional method of characteristics, the Numerical Laplace Transform and an experimental measurement.

4.3.1 Sagging between 2 towers

A single line 600m long with a sagging between towers is analyzed. The line maximum and minimum heights are 28m at the towers and 8m at the middle span. A unit step voltage source is connected to the sending node, while the receiving node is left open. Frequency dependence of the line parameters is accounted for. Fig. 5 shows the voltage at the receiving end of the line, comparing the results obtained with the Numerical Laplace Transform (NLT) [14, 20-21] and the simplified method of characteristics (SMC). Results when the line presents no sagging (UL) are also included.

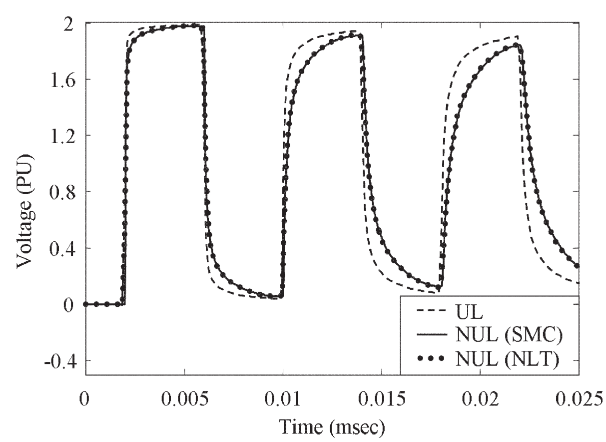

Fig. 5. Voltage at the receiving end of the non uniform line. Simplified method of characteristics (SMC) vs. Numerical Laplace Transform (NLT).

In Fig. 6, a comparison between simplified and conventional methods of characteristics (SMC and MC, respectively) is provided. Results are virtually identical, but the simulation with the simplified method of characteristics was several times faster than the conventional one; between 5 and 6 times faster using MATLAB® 7.

4.3.2 Simulation of a field experiment

As a second example, the proposed method is applied to the simulation of a field experiment performed by Wagner, et al. [22]. The experiment consists on injecting a step like wave at

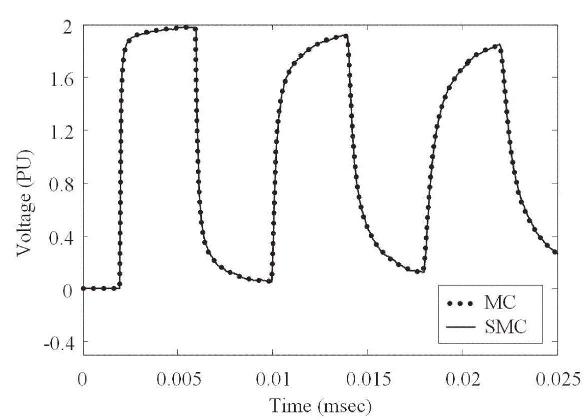

Fig. 6. Voltage at the receiving end of the non uniform line. Conventional method of characteristics (MC) vs. simplified method of characteristics (SMC).

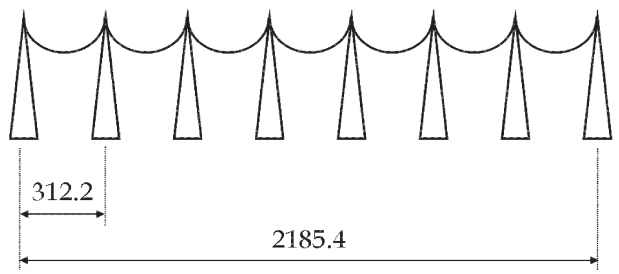

Fig. 7. Simplified characteristics grid.

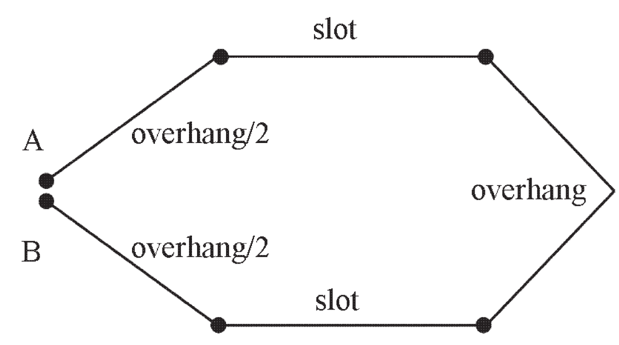

Fig. 9. Winding representation using line segments.

one end of a 2185.4m long line divided in 7 equal segments, as shown in Fig. 7. The line maximum and minimum heights are 26.2m at the towers and 15.24m at the middle span. The 3 line conductors are ACSR with radius of 2.54cm. The injected wave is applied simultaneously to the 3 conductors at the sending node, while the receiving node is left open.

For the simulation, the line is represented by a single-phase equivalent. The voltage at the receiving end of the line is shown in Fig. 8, comparing the experimental results and those obtained with the simplified method of characteristics. Waveforms were plotted as half of their actual magnitude, as done in [22], to remove the doubling due to the open circuit.

4.3.3 Step response of a machine winding (single coil)

As a last example, the step response of a machine winding is analyzed using a non uniform line model. Electrical parameters of the winding coil are computed for 2 different regions, namely the slot and the overhang, according to [23] and the data listed in Table 1. The coil is divided in 5 segments, as shown in Fig. 9. The complete coil model is obtained by means of the cascaded connection of 5 different chain matrices. A unit step voltage source is connected to node A, while node B is left open.

In Fig. 10, the voltage waveform at node B is shown, comparing the results from the simplified method of characteristics from those obtained with the Numerical Laplace Transform algorithm.

5. Conclusions

In this work, two major improvements to the method of characteristics have been presented, resulting in the simplified method of characteristics (SMC) for the solution of symmetrical

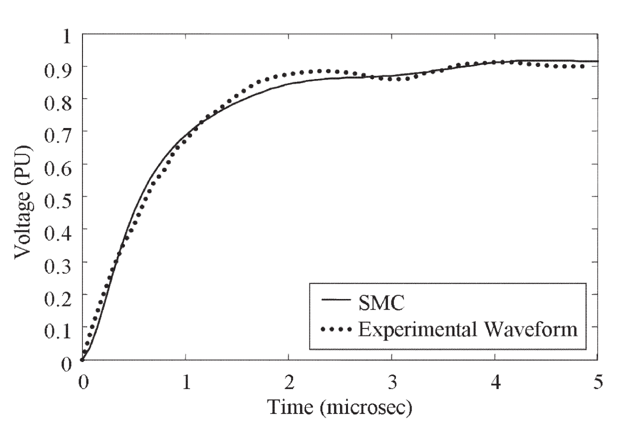

Fig. 8. Simplified characteristics grid.

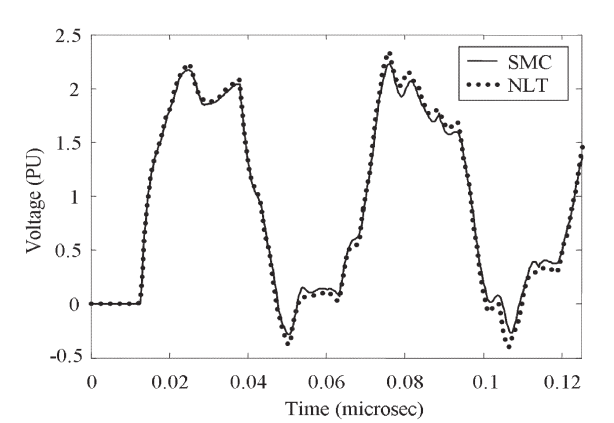

Fig. 10. Voltage at node B of the winding.

Table. 1. Winding data.

Slot width	0.75 m
Slot material	Iron
Turn area	3 x 9 mm
Turn length	3.8 m
Slot length	0.75 m
Overhang length	1.15 m
Conductor material	Cupper

non uniform lines with frequency dependence. This method does not require the interpolation process previously needed to obtain a regular grid of characteristics for the non uniform line. Instead, a uniform line equivalent to the non uniform line is obtained, for which the conventional solution stands. Besides, a new solution procedure has been defined, for which the spatial discretization is not necessary, turning the algorithm faster and simpler. Accuracy of the proposed method has been tested by means of 3 application examples, providing comparisons with a frequency domain method (NLT) and an experimental result.

6. References

[1] E.A. Oufi, A.S. Alfuhaid, and M.M. Saied, «Transient analysis of lossless single-phase nonuniform transmission lines,» *IEEE Trans. Power Delivery*, vol. 9, pp. 1694–1700, July 1994.

[2] M.S. Mamis and M. Koksal, «Lightning Surge Analysis using Non-uniform, Single-phase Line Model», *IEE Proc. Gener. Transm. Distrib.*, vol. 148, no. 1, pp. 85-90, January 2001

[3] P. Gómez, P. Moreno, J.L. Naredo, «Frequency Domain Transient Analysis of Non-Uniform Lines with Incident Field Excitation», *IEEE Trans. Power Delivery.*, vol. 20, no. 3, pp. 2273-2280, Julio 2005.

[4] M. Davila, J.L. Naredo, P. Moreno, «Transient Analysis of Power Lines Including Nonuniform and Frequency Dependent Effects», *Proc of the 34th North American Power Symposium (NAPS'02)*, Tempe, AZ, U.S.A., October 14-15, 2002.

[5] A.R. Chávez, P. Moreno, J.L. Naredo, L. Guardado, ''Fast Transients Analysis of Non-Uniform Multiconductor Frequency-Dependent Transmission Lines», *IEEE Trans. Power Delivery*, vol. 21, no. 2, April 2006.

[6] C. Menemenlis and Z.T. Chun, «Wave propagation on nonuniform lines,» *IEEE Trans. Power Apparat. Syst.*, vol. PAS-101, pp. 833–839, Apr. 1982.

[7] H.V. Nguyen, H.W. Dommel, and J.R. Marti, «Modeling of singlephase nonuniform transmission lines in electromagnetic transient simulations,» *IEEE Trans. Power Delivery*, vol. 12, pp. 916–921, Apr. 1997.

[8] M.T.C. de Barros and M.E. Almeida, «Computation of electromagnetic transients on nonuniform transmission lines,» *IEEE Trans. Power Delivery*, vol. 11, pp. 1082–1091, Apr. 1996.

[9] A. Ramirez, A. Semlyen, and R. Iravani, «Modeling nonuniform transmission lines for time domain simulation of electromagnetic transients,» *IEEE Trans. Power Delivery*, vol. 18, No. 3, pp. 968–974, July 2003.

[10] J.A. Gutierrez, P. Moreno, J.L. Naredo, J.L. Bermúdez, M. Paolone, C.A. Nucci, F. Rachidi,» Nonuniform Transmission Tower Model For Lightning Transients Studies», *IEEE Trans. Power Delivery*, vol. 19, no. 2, pp. 490-496, April 2004.

[11] P. Gómez, P. Moreno, J.L. Naredo, M. Dávila, «Frequency Domain Transient Analysis of Transmission Networks Including Non-linear Conditions», *Proc. of IEEE Bologna Power Tech International Conference (BPT'03)*, June 2003, Bologna, Italy.

[12] M. Dávila, J.L. Naredo, P. Moreno, A. Ramírez, « Practical Implementation of a Transmission Line Model for Transient Analysis Considering Corona and Skin Effects», *IEEE Bologna Power Tech International Conference*, June 23-26, 2003, Bologna Italy.

[13] A. Ramírez, J.L. Naredo, P. Moreno, L. Guardado, «Electromagnetic Transients in Overhead Lines Considering Frequency Dependence and Corona Effect via the Method of Characteristics», *Int. Journal of Electrical Power and Energy Systems*, vol. 23, pp. 179-188, 2001.

[14] J.A. Gutierrez, P. Moreno, J.L. Naredo, J.C. Gutierrez, «Fast Transients Analysis of Nonuniform Transmission Lines Through the Method of Characteristics», *Int. Journal of Electrical Power and Energy Systems*, vol. 24, pp. 781-788, 2002.

[15] A. Ramirez, J.L. Naredo and P. Moreno, «Full Frequency-dependent Line Model for Electromagnetic Transient Simulation Including Lumped and Distributed Sources», *IEEE Trans. Power Delivery*, vol. 20, no. 1, pp. 292-299, Jan. 2005.

[16] R. Radulet, A. Timotin, A. Tugulea, A. Nica, «The Transient Response of Electric Lines Based on the Equations with Transient Line-parameters», *Revue Roumaine de Science et Technologie-Electrote-chn. et Energ*. Vol.23, no. 1, pp.3-19, 1978.

[17] A. Semlyen, A. Dabuleanu, «Fast and Accurate Switching Transient Calculations on Transmission Lines with ground return using recursive convolutions», *IEEE Trans. Power Apparatus and Systems*, vol. PAS-94, pp. 561-571, March/April 1975.

[18] R. Courant, K. Friedrichs and H. Lewy, «Über die partiellen Differenzengleichungen der mathematischen Physik»,

Mathematische Annalen, vol. 100, no. 1, pages 32–74, 1928.
[19] C.R. Paul, *Analysis of Multiconductor Transmission Lines*, John Wiley & Sons, U.S.A., 1994
[20] D.J. Wilcox, «Numerical Laplace Transformation and Inversion», *Int. J. Elect. Enging. Educ.*, vol 15, pp. 247-265, 1978.
[21] L. M. Wedepohl, «Power System Transients: Errors Incurred in the Numerical Inversion of the Laplace Transform», *Proc. of the 26th Midwest Symposium on Circuits and Systems*, August 1983.
[22] C.F. Wagner, I.W. Gross, and B.L. Lloyd, «High-voltage impulse test on transmission lines,» *Amer. Inst. Elect. Eng. Trans.*, pt. III-A, vol. 73, pp. 196–210, Apr. 1954.
[23] J.L. Guardado, K.J. Cornick, «Calculation of Machine Winding Electrical Parameters at High Frequencies for Switching Transient Studies», *IEEE Trans. on Energy Conversion*, vol. 11, No. 1, March 1996.
[24] B. Gustavsen and A. Semlyen, «Rational approximation of frequency domain responses by Vector Fitting», *IEEE Trans. Power Delivery*, vol. 14, no. 3, July 1999, pp. 1052-1061.

Appendix: Recursive convolution algorithm

Convolution term from (4a) is defined in time domain as

$$\psi = -\sum_{i=1}^{N_1} k_i p_i \int_0^t e^{-p(t-\tau)} i(\tau) d\tau \qquad (29)$$

Due to the non-smooth nature of $R'(s)$, complex conjugate pairs of poles and residues are used in the rational approximation of the Laplace domain spectrum of [24]:

$$\Psi(s) = -\sum_{i=1}^{N_1} \Psi_i, \qquad i = 1,2,\ldots,N_1 \qquad (30)$$

where N_1 is the order of the approximation and

$$\Psi_i = \left[\frac{a_i s + b_i c_i}{s^2 + d_i s + b_i}\right] I(s) \qquad (31)$$

Coefficients a_i, b_i, c_i and d_i are always real. Eq. (31) can be written in time domain as

$$\frac{d^2\psi_i}{dt^2} + d_i \frac{d\psi_i}{dt} + b_i \psi_i = a_i \frac{d\,i(t)}{dt} + b_i c_i i(t) \qquad (32)$$

Applying the central differences rule to (32) and in accordance to (30), the total convolution can be written as follows:

$$\psi_{n+1} = -\sum_{i=1}^{N_1} \frac{\Delta t}{1 + d_i \Delta t/2} \left[\frac{a_i}{2}(i_{n+1} - i_{n-1}) + b_i c_i \Delta t i_n + \frac{\psi_{i,n}(2 - b_i \Delta t^2) - \psi_{i,n-1}(1 - d_i \Delta t/2)}{\Delta t} \right] \qquad (33)$$

A similar procedure can be applied to the convolution term ϕ related to (4b), yielding:

$$\phi_{n+1} = -\sum_{i=1}^{N_2} \frac{\Delta t}{1 + h_i \Delta t/2} \left[\frac{e_i}{2}(v_{n+1} - v_{n-1}) + f_i g_i \Delta t v_n + \frac{\phi_{i,n}(2 - f_i \Delta t^2) - \phi_{i,n-1}(1 - h_i \Delta t/2)}{\Delta t} \right] \qquad (34)$$

where coefficients e_i, f_i, g_i and h_i are all real and computed from the rational fitting of ϕ in the Laplace domain.