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Boundary Element Analysis of Elasto-Plastic Problems

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1. Abstract

This paper presents the application of the Boundary Element Method for nonlinear time-independent problems. Part of the domain, where the plastic phenomena takes place is discretized into quadratic, quadrilateral, continuous internal cells, in order to obtain the plastic strain. The plastic analysis is basically applied to metals. The Von Mises yield criterion and strain hardening are used in this analysis. Numerical results are compared with the solution obtained from the Finite Element Method (FEM) and relevant references.

Key words: plasticity, boundary element method

2. Resumen (Análisis de elementos de frontera en problemas elastoplásticos)

Este artículo presenta la aplicación del *método de elementos de frontera* a problemas no lineales independientes del tiempo. Parte del dominio, especialmente la parte más crítica donde es más susceptible a la cedencia, en la cual se genera la plasticidad, es dicretizada con celdas internas cuadriláteras cuadráticas continuas para obtener la deformación plástica. El análisis plástico es básicamente aplicado a metales en este trabajo. El criterio de cedencia de von Mises y el endurecimiento por deformación son considerados en este análisis. Resultados numéricos son comparados con soluciones obtenidas del *método de elementos finitos* y referencias.

Palabras clave: plasticidad, elementos de frontera.

3. Introduction

Most of the materials used in engineering have sophisticated material properties which may depend on stress, time and temperature. In order to model the complex behaviour of such materials, stress analysis techniques are developed. These techniques are necessary to solve the elastic problem but also go further to model the non-elastic phenomenon such as plasticity.

For many years problems of stress analysis in industry have been solved using Finite Difference Method and the Finite Element Method (FEM). An alternative method to these domain type methods is the Boundary Element Method (BEM). Despite this, it still takes considerable time to perform in a particular plasticity analysis.

During many years the FEM has been used as the main tool to solve problems in engineering [1]. The domain of the body is divided into several small sub domains, of quite simple shape, called finite elements. Any continuous parameter such as pressure or displacement can be approximated to the actual behaviour of the solution with trial functions, usually polynomials. These functions are uniquely defined in terms of the approximated values of the solution at some nodal points, inside or on the boundary of each element.

A weighted residual technique is the most popular tool to assess this approximation, leading to a symmetric system of equations which involves the unknown values of the approximated solution at nodal points. Without any doubt, this method is computationally efficient and during many years has reached such popularity that a very wide range of linear and non-linear engineering problems have been solved with this powerful numerical method [2].

The Boundary Element Method (BEM) is a less mature technique but has reached a level of development in certain fields that has become an essential tool for design engineers. The BEM has many applications also but not as many as FEM. Nevertheless this method is an effective alternative to FEM in many important areas of engineering analysis. The BEM is a relatively new technique for engineering analysis;

the fundamental can be traced back to mathematical formulations by Fredholm [3] and Mikhilin [4] in potential theory and Betti [5], Somigliana [6] and Kupradze [7] in elasticity. In the context of the BEM, also called Boundary Integral Equation (BIE) [8]], the formulations are due to Jaswon [9], Hess and Smith [10], Massonet [11], Rizzo [12] and Cruse [13]. But perhaps the most significant early contribution to BEM as an effective numerical technique is due to the work developed by Lachat [14] and Lachat and Watson [15]. They developed an isoparametric formulation similar to the FEM and proved that the BEM can be used as an efficient tool for solving problems with sophisticated configurations. As an application example, Urriolagoitia *et al*. [16,17] followed a similar approach in order to determine direction of crack propagation, considering the variation of the specimen geometry, as well as, different combinations of biaxial loading applied on the boundary of the specimen as fundamental parameters.

The reduction of the dimensionality of the problem is one of the most important attractive features of this technique; in the two-dimensional case only the boundary of the domain needs to be discretized and for three-dimensional problems the surface is discretized into a number of boundary elements over which polynomial functions, of the type used in finite elements, are introduced to interpolate the values of the approximated solution between the nodal points. Following discretization of the boundary and the evaluation of the relevant integrals, a matrix system of equations is obtained, which, is fully populated and non-symmetric, is of much smaller size than the FEM.

Some of the main characteristics of BEM are: *i*) reduced set of equations, *ii*) simple data preparation, *iii*) semi-infinite or infinite boundaries need not be accurate modeled, *iv*) accurate selective calculation of internal stresses and *v*) displacements and great resolution for stress concentration problem. These features plainly justify the increasing popularity achieved in recent years. BEM methods can be classified into two groups indirect and direct formulation.

One of the first successful applications of the BEM to nonlinear problems in solid mechanics with the formulation for the time-independent plasticity was due to Swedlow and Cruse [18]. This was followed by a numerical implementation by Riccardella [19]. Another formulation for plasticity based on initial stress is due to Banerjee and Mustoe [20].

Two examples are considered in this paper; the first one is a perforated aluminum plate with tensile load and the second one is a thick pressurized cylinder in order to determine the plastic front.

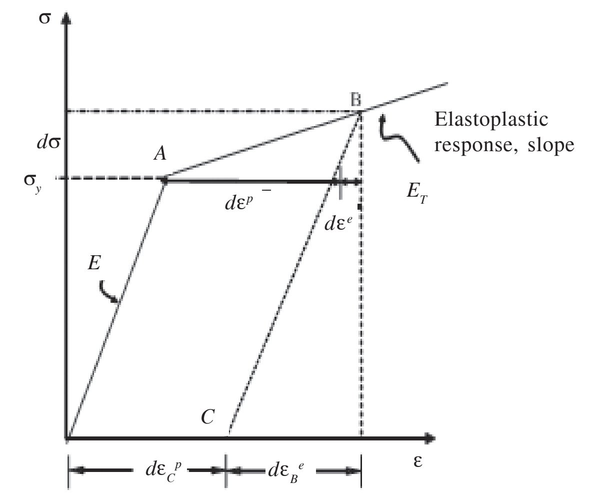

Fig. 1. Elasto-Plastic behaviour for a linear strain hardening.

4. Theoretical concepts

Introduction to Elasto-Plasticity

In any material, there is an elastic range in which the limit is the yield stress σ_y. After this limit, when a certain level of stress has been reached, a plastic deformation occurs. By considering the case of a body subjected to uniaxial loading (simple tension or compression test), there are two possible material responses; elastic-perfect plastic (without hardening) and elasto-plastic (with hardening) see figure 1. At some point after the yield stress, consider a further load producing an increment of stress, $d\sigma$ which generate a change of strain, $d\varepsilon$. Since the strain can be separated into elastic and plastic parts, so that

$$d\varepsilon = d\varepsilon^e + d\varepsilon^p \tag{1}$$

After plastic deformation occurs the material has the ability to endure a greater stress due the hardening, which is presented in figure 1. After the material has yielded the plastic strain may increase even if the load is decreased as it was shown by Urriolagoitia-Sosa *et al*. [21] and López-Castro *et al*. [22].

Figure 1 shows that during loading and unloading the behaviour of the material is different. The loading is done in an elastic and plastic way but the unloading in an elastic way.

In linear hardening, during the load, after yield stress in tension, the material reaches point *B* in figure 1, $d\varepsilon^p$ and $d\varepsilon^e$ denote the increment of the plastic and elastic component of the strain respectively. If the load is completely removed, point *C* on the strain axis is reached and the elastic strain at this point is zero and the increment of the plastic strain $d\varepsilon_C^{\,p} = d\varepsilon_B^{\,p}$. In order to increase the plastic strain it is necessary a stress beyond its previous value σ_B, which is known as the subsequent yield stress and it changes as the plastic strain changes. A yielding in compression occurs if the load is inverted; at this point this kind of yielding depends on the type of hardening behaviour. According to the experiments, the compressive yield stress changes depending on the previous deformation history. Some alternative models which describe the strain hardening are:

Isotropic hardening. The subsequent yield stresses in tension and compression are equals and the yield stress has the same behaviour. In this model the compressive yield stress does not change with the previous deformation history.

Cinematic hardening. This type of hardening is said to take place when the elastic range is preserved during the process. In other words the material experiences a rigid body motion. This means the difference between yielding in tension and in compression is equal to the initial yield stress difference. Such a hardening model gives rise to the experimentally observed Bauschinger effect.

Independent hardening (mixed). The behaviour of hardening is independent in tension and compression. This is the more general rule.

Elasto-Plastic Stress-Strain Relationship

During any increment of the stress, after the initial yielding, the material behaviour is divided into elastic and plastic parts. By considering the case of a uniaxial load with linear hardening, the total strain is expressed as follows:

$$\varepsilon = \varepsilon^e + \varepsilon^p \tag{2}$$

$$\varepsilon^e = \sigma/E \tag{3}$$

Where E is the modulus of elasticity, ε^e is the elastic part of the strain tensor and ε^p is the plastic part.

Once the applied stress has passed the yield stress, stresses and strains for loading in tension are related

$$\sigma_t = E_T\varepsilon^e \tag{4}$$

From the curve stress-strain in figure 1, it is clear that $E\varepsilon^e$ can be replaced for $E_T\varepsilon$ so,

$$\sigma_t = E_T\varepsilon + \sigma_y \tag{5}$$

Where E_T is the tangential modulus of elasticity and equation (5) is for a level of stress greater than the yield stress. Since the total strain is divided in the elastic and plastic component according to (2), equation (5) will be

$$\sigma_t = E_T(\varepsilon^e + \varepsilon^p)\sigma_y \tag{6}$$

From figure 1 it is possible to write the strain-hardening parameter in terms of increments as

$$H' = \frac{d\sigma}{d\varepsilon^p} = \frac{d\sigma}{d\varepsilon - d\varepsilon^p} \tag{7}$$

by substituting $d\varepsilon^e = \dfrac{d\sigma}{E}$ and after some algebraic steps it is possible to obtain the hardening as

$$H' = \frac{E_T}{1 - E_T/E} \tag{8}$$

The final result for the stress after yielding is

$$\sigma_t = H'\varepsilon^p + \sigma_y \tag{9}$$

where H' can be explained as the slope of the strain-hardening part of the stress-strain curve after removal of the elastic strain component. The term $d\sigma$ can be interpreted as the stress increment necessary to cause the strain increment $d\varepsilon^p$.

Inside the plastic range to start the yielding it is necessary to go through the yield surface. There are two hypotheses to compute this limit: the strain hardening and the work hardening. The strain hardening assumes the hardening depends on plastic deformation. The work hardening supposes that the yield surface depends only on the total plastic work and is a function of a hardening parameter h, which for linear hardening is equivalent to H', that is

$$h = W^p = \int \sigma d\varepsilon^p \tag{10}$$

which is called total dissipation of the energy and represents the total energy exchange that has occurred in the permanent deformation process, and is often used to characterize hardening. This equation geometrically represents the area below the stress-strain curve corresponding to the plastic component. Hence

$$\sigma_t(h) = \sigma_t \int \sigma d\varepsilon^p \tag{11}$$

The yield function for this case can be written as

$$f(\sigma,h) = F(\sigma) - \sigma_t(h) = 0 \tag{12}$$

Where σ is the current stress, σ_t is the current yield stress and H' is the hardening parameter which governs the expansion of the yield stress.

The von Mises yield criterion

This criterion states that a material will yield when the deviatoric stress tensor J reaches some critical value. Since only the deviatoric stress tensor contributes to plasticity, the yield stress can be written in terms of the invariants of the state of stresses J_2 and J_3.

In order to define J_2 and J_3 it is necessary to compute the mean normal stress as:

$$\sigma_m = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \tag{13}$$

where σ_1, σ_2 and σ_3 are the principal stresses. The principal state of stress can be formed by the superposition of two principal stress states. The first one is the mean normal stress and the second one represents the deviation of the original state of stress. This decomposition is

$$\sigma_{ij} = \sigma_m + S_{ij} \tag{14}$$

where σ_{ij} is the original state of stress and S_{ij} is the deviation or deviatoric stress of the original state.

The invariants J_2 and J_3 can be written in terms of the deviatoric stresses as

$$J_2 = \frac{1}{2} S_{ij} S_{ij} \tag{15}$$

and

$$J_3 = \frac{1}{3} S_{ij} S_{ij} S_{ij} \tag{16}$$

therefore the yield function for multiaxial stresses state, which is a more general function, is

$$f(J_2, J_3, h) = F(\sigma_{ij}) - \sigma_t(h) = 0 \tag{17}$$

This criterion, von Mises, does not depend on J_3, only the second invariant of stress deviations, J_2, contributes to plasticity. Thus equation (17) can be rewritten as:

$$f(J_2, h) = \sqrt{3J_2} - \sigma_t(h) = 0 \tag{18}$$

In terms of loading it is possible to define the yield function. The incremental change in the yield function due to an incremental stress change is

$$df\left(\sigma_{ij}\right) = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} \tag{19}$$

In this equation $\frac{\partial f}{\partial \sigma_{ij}}$ is the rate of change of the yield surface with respect to the principal stresses direction and $d\sigma_{ij}$ is the magnitude of the increment of the stress.

If

$d\sigma_{ij} = 0$, the case is perfectly plastic and the stress point remains on the yield surface

$d\sigma_{ij} < 0$, an elastic unloading takes place and the stress point returns inside the yield surface

$d\sigma_{ij} > 0$, there is a linear strain hardening, a plastic deformation occurs and the stress point grows beyond the yield surface.

Prandtl Reuss Equations

In order to derive the relationship between the plastic strain component and the stress increment it is convenient to assume that the plastic strain increment is proportional to the variation of the stress, so that

$$d\varepsilon^p = \frac{\partial f}{\partial \sigma_{ij}} d\lambda \tag{20}$$

This equation is called the normality condition because

$\frac{\partial f}{\partial \sigma_{ij}}$ is a vector directed normal to the yield surface at the stress point, see figure 2, and $d\lambda$ is a proportionality constant termed plastic multiplier which may vary throughout the loading history.

The equation (20) is named flow rule or plastic flow equation which involves the magnitude and direction of the components of $d\varepsilon^p$.

By relating the von Mises criterion and the flow rule, the Prandtl Reuss equations are obtained and these can be written as

$$d\varepsilon^p = \frac{\partial f}{\partial \sigma_{ij}} d\lambda = S_{ij} d\lambda \tag{21}$$

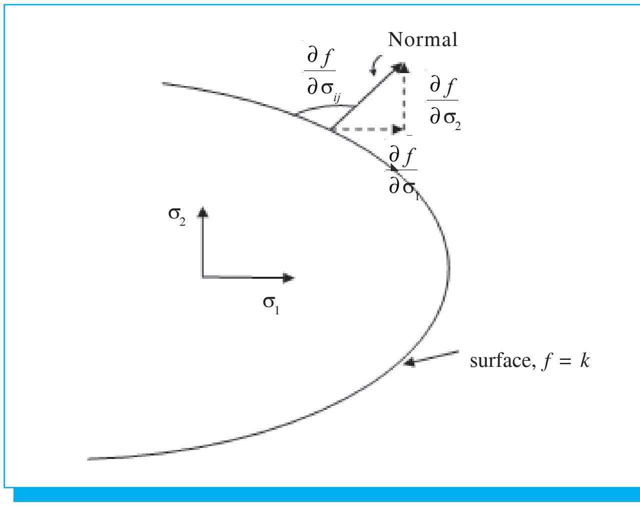

Fig. 2. Vector directed normal to the yield surface.

Equivalent Stress and Plastic Strain

Since the material properties such as the hardening parameters and the yield stress are obtained from uniaxial loading tests, it is necessary to state a correlation between them and multiaxial stress state. These can be through the equivalent quantities namely: the equivalent or effective stress. The equivalent or effective stress can be defined as

$$\sigma_{eq} = \sqrt{3J_2} = \sqrt{\frac{3}{2} S_{ij} S_{ij}} \tag{22}$$

and from the dissipation total of the energy it is possible to obtain the plastic deformation equivalent as

$$d\varepsilon_{eq}^{p} = \sqrt{\frac{2}{3} d\varepsilon_{ij}^{p} d\varepsilon_{ij}^{p}} \tag{23}$$

by expressing the Prandtl Reuss equations in terms of σ_{eq} and $d\varepsilon_{eq}^{p}$ we have

$$d\varepsilon_{ij}^{p} = \frac{3}{2} \frac{d\varepsilon_{eq}^{p}}{\sigma_{eq}} S_{ij} \tag{24}$$

In order to solve an elasto-plastic problem the integration over the loading history is required. Since the equations above are of an incremental nature involving plastic strain related to increments of stress or strain, are the necessary functions to solve elasto-plastic problems.

Total Strains Related to Prandtl Reuss Equations

It is possible to relate the Prandtl Reuss equations with a new variable called total strain, in order to compute the plastic strain increments. The procedure can be as follows: let us say we have reached a given state of stresses and accumulated plastic strain in some point of the loading path. By increasing the load a small amount there is an additional plastic strain produced and the total strain is given by

$$\varepsilon_{ij}^{T} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p} + \Delta\varepsilon_{ij}^{p} \tag{25}$$

where ε_{ij}^{e} is the elastic part of the total strain with the load increment, ε_{ij}^{p} is the plastic strain without the load increment and $\Delta\varepsilon_{ij}^{p}$ is the plastic strain increment produced by the increment of the load.

By substituting the Hooke's law into elastic strain component, the modified total strain can be written as

$$\varepsilon' = \frac{1}{2G}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}\right) + \Delta\varepsilon_{ij}^{\ p} \tag{26}$$

The Prandtl Reuss equation in terms of the increments $\Delta\varepsilon_{ij}^{p}$ and $\Delta\lambda$ can be modified as

$$\varepsilon' = \Delta\varepsilon_{ij}^{\ p}\left(\frac{1}{2G\Delta\lambda} + 1\right) \tag{27}$$

It is convenient to use equivalent quantities, in order to write the modified total strain in these terms, which is

$$\varepsilon'_{eq} = \sqrt{\frac{2}{3} e'_{ij}\, e'_{ij}} \tag{28}$$

A very important elasto-plastic relationship between total strain and the increment of the plastic strain is

$$\Delta\varepsilon_{ij}^{\ p} = \frac{3G\varepsilon'_{eq} - \sigma_{eq,i-1}}{3G + H'_{i-1}} \tag{29}$$

where G is the shear modulus and H'_{i-1} is the strain hardening or plastic modulus calculated before the load increment. The equation above computes the increment of the plastic deformation for every step in the incremental loading process.

5. Methodology

Elastoplastic formulation for BEM

Plasticity is considered a time-independent phenomenon, but the equations are associated to a loading factor which is like a time parameter. Because of this, the rate notation is used in this work. The rates used here symbolize the current value of the variable and they have nothing to do with the derivative with respect to time or space, these rates are the derivative with respect to a loading factor.

The equilibrium conditions that must be satisfied on the domain, can be represented in terms of rates as follows

$$\dot{\sigma}_{ij,i} + \dot{b}_j = 0 \tag{30}$$

and on the boundary

$$\dot{t} - \dot{\sigma}_{ij}\eta_j = 0 \tag{31}$$

where $\dot{b}_j$ are the body forces and η_j are the components of the outward normal to the boundary.

In terms of displacements Navier's equations for elastoplasticity can be developed as in elasticity, so the governing differential equations of the problem are obtained, but now the rate form of the equations instead. The substitution of the relationship between the stress and the strain rates in terms of displacements into the equilibrium equation gives

$$\left(2\mu\dot{\varepsilon}_{ij} + \frac{2\mu\nu}{1-2\nu}\dot{\varepsilon}_{dd}\delta_{ij} - \dot{\sigma}'_{ij}\right)_{,i} + \dot{b}_j = 0 \tag{32}$$

to obtain

$$\mu\dot{u}_{i,ij} + \mu\left(\frac{1}{1-2\nu}\right)\dot{u}_{j,ji} - 2\mu\dot{\varepsilon}'_{ij,i} - \frac{2\mu\nu}{1-2\nu}e_{,j}\delta_{ij} + \dot{b}_j = 0 \tag{33}$$

the equation (32) is for internal points, but boundary conditions must be also satisfied. The boundary conditions in terms of rates are; for displacements $\dot{u}_i = \ddot{u}_i$ and for tractions $\dot{t}_i = \ddot{t}_i$ and the equation representing the traction boundary conditions is,

$$\dot{t} - \left(2\mu\dot{\varepsilon}_{ij} + \frac{2\mu\nu}{1-2\nu}\dot{\varepsilon}_{dd}\delta_{ij} - \dot{\sigma}'_{ij}\right)n_j = 0 \tag{34}$$

to obtain

$$\dot{t} + 2\mu\left(\dot{\varepsilon}'_{ij}n_j + \frac{\nu}{1-2\nu}en_i\right) = \frac{2\mu\nu}{1-2\nu}\dot{u}_{j,j}n_i + \mu\left(\dot{u}_{i,j} + \dot{u}_{j,i}\right) \tag{35}$$

$$\mu\dot{u}_{i,ij} + \mu\left(\frac{1}{1-2\nu}\right)\dot{u}_{j,ji} - 2\mu\dot{\varepsilon}'_{ij,i} - \frac{2\mu\nu}{1-2\nu}e_{,j}\delta_{ij} + \dot{b}_j = 0 \tag{36}$$

These equations (34), (35) and (36) are for three dimensional problems. In order to work with two dimensional problems for the plane stress state it is necessary to remove the strain in *z* direction, so $\dot{\varepsilon}'_{33} = 0$.

So far the nonlinear problem has been analyzed, which means that it is not possible to solve the resulting governing equations directly like in elasticity. It is possible to solve the nonlinear elastoplastic problem by using a method which involves essentially the solution of an elastic problem in each iteration, this method is called *successive elastic solution* and it is used in this work.

Integrating by parts, the integral involving the symmetry of Hooke's law in the domain Ω′ without the integration of the initial strain term, the following expression is obtained:

$$\int_\Omega \sigma'_{ij,j}\dot{u}_i d\Omega + \int_\Gamma \sigma'_{ij} n_j \dot{u}_i d\Gamma - \int_\Omega \sigma'_{ij}\dot{\varepsilon}^a_{ij} d\Omega = \int_\Omega \dot{\sigma}_{ij,j}\dot{u}'_i d\Omega + \int_\Gamma \dot{\sigma}_{ij} n_j \dot{u}'_i d\Gamma \tag{37}$$

The equilibrium equations and traction definition can be substituted into equation above to obtain

$$-\int_\Omega \dot{b}_i\dot{u}_i d\Omega + \int_\Gamma t'_i\dot{u}_j d\Gamma - \int_\Omega \sigma'_{ij}\dot{\varepsilon}^a_{ij} d\Omega = -\int_\Omega \dot{b}_i u'_j d\Omega + \int_\Gamma \dot{t}_i u'_i d\Gamma \tag{38}$$

Where $\dot{u}_i$, $\dot{t}_i$, $\dot{\sigma}_{ij}$ y $\dot{\varepsilon}_{ij}$ are the displacement, traction, stress and strain rates respectively which belong to the domain Ω enclosing Ω′. This leads to the following boundary integral representation of the boundary displacements when the initial strain approach for the solution of elastoplastic problems

$$c_{ij}\dot{u}_i + \oint_\Gamma t'_{ij}\dot{u}_j d\Gamma = \int_\Gamma u'_{ij}\dot{t}_j d\Gamma + \int_\Omega \sigma'_{ijk}\dot{\varepsilon}_{jk}{}^a d\Omega \tag{39}$$

In a similar way, the boundary integral equation of the internal stresses is expressed by

$$\dot{\sigma}_{ij} = \int_\Gamma D_{ijk}\dot{t}_j d\Gamma - \int_\Gamma S_{ijk}\dot{u}_j d\Gamma + \oint_\Omega \Sigma_{ijk}\dot{\varepsilon}^a{}_{jk} d\Omega + f_{ij}\dot{\varepsilon}^a{}_{jk} \tag{40}$$

Where $\oint$ is a Cauchy integral, S_{ij} are terms containing the derivative of the displacements and tractions, f_{ij} is the free term and Σ_{ij} is the fundamental solution for the domain.

The solution of Navier's differential equation through the use of the Galerking vector it is called the fundamental solution for a unit force point applied to the body at point *d*.

The displacement and tractions fundamental solutions for the displacement boundary equation in the two-dimensional planes are

$$\dot{u}'_i = \frac{1}{8\pi\mu(1-\nu)}\left[(3-4\nu)\ln\left(\frac{1}{r}\right)\delta_{ij} + r_{,i}r,j\right] \quad (41)$$

$$\dot{t}'_{ij} = \frac{-1}{4\pi(1-\nu)r}\{[(1-2\nu)\delta_{ij} + 2r_{,i}r_{,j}]\frac{\partial r}{\partial n} -$$

$$-(1-2\nu)(r_{,i}n_j - r_{,j}n_i)\} \quad (42)$$

$$\sigma'_{ijk} = \frac{-1}{4\pi(1-\bar{\nu})r}\{((1-2\nu)(r_{,j}\delta_{ki} + r_{,i}\delta_{jk} - r_{,k}\delta_{ij})) + \}$$

$$+2r_{,i}r_{,j}r_{,k} \quad (43)$$

Numerical Integration

The domain Ω_Y is divided in N_C cells as follows

$$\Omega_Y = \overset{N_c}{\underset{n=1}{U}}\Omega_n \quad (44)$$

The plastic terms for the strain and stress rate tensors are given, at every cell Ω_n, by

$$\dot{\varepsilon}^a_{ij} = \sum_{L=1}^{n_c}\Psi_L\dot{\varepsilon}^{a,k}_{ij} \quad (45)$$

$$\dot{\sigma}^a_i = \sum_{L=1}^{N_c}\Psi_L\dot{\sigma}^{a,k}_{ij} \quad (46)$$

where n_c is the number of nodes in the cell, N_C is the number of cells and Ψ_L are the shape functions. The numerical expression for the displacement on the boundary is

$$c\dot{u} + \sum_{n=1}^{N_{el}}\left(\int_\Gamma T\phi d\Gamma\right)\dot{u}^n = \sum_{n=1}^{N_{el}}\left(\int_\Gamma U\phi d\Gamma\right)\dot{t}^n +$$

$$+\sum_{n=1}^{N_{el}}\left(\int_{\Omega_N}\sigma\Psi d\Omega\,\dot{\varepsilon}^{g,n}\right) \quad (47)$$

The terms *T*, *U* and σ in this equation, are sub matrices containing the fundamental solution. N_{el} is the number of integration elements. Similarly to the boundary, the discretized expression for the domain stresses can be obtained

$$\dot{\sigma}_{ij} = \sum_{n=1}^{N_{el}}\int_\Gamma D\dot{t}d] - \sum_{n=1}^{N_{el}}\int_\Gamma S\dot{u}d\Gamma + \sum_{n=1}^{N_{el}}\int_\Omega\Sigma\Psi d\Omega\dot{\varepsilon}^{g,n} +$$

$$+ f_{ij}(\dot{\varepsilon}^g) \quad (48)$$

The quantities *D*, *S* and Σ are sub matrices containing the derivative of the fundamental solution and Ψ are the shape functions corresponding to the boundary elements and cells respectively.

6. Results

Benchmark Problems

To outline the applicability of the formulation described in the previous sections some examples were run in a Fortran code written by Leitao and the results have been compared with other references and (FEM).

The Perforated Plate

A perforated and aluminum plate is presented in this first problem, its geometry is presented in Figure 3. The boundary of the problem is discretized with quadratic elements and the domain with interior quadratic cells. The problem has been analyzed experimentally by Theocaris and Marketos [23]. The material is considered with linear hardening and has the following properties: Young's Modulus E = 7 000 kg/mm², Poisson's ratio ν = 0.2, Hardening Coefficient or Plastic Modulus H' = 224 kg/mm², and the Yield Stress σ_y = 24.3 kg/mm². The Von Mises Yield Criterion was used in all the problems.

Figure 5 presents the results for the equivalent strains (in every step of the load), normalized with respect to the yield stress, at root of plate (point of maximum stress). Here *E* is the modulus of elasticity and λ is the load factor. It can be seen the convergence of these curves for different increments of the load factor. Figure 6 exhibits the results for variation of the mean stress, normalized with the yield stress, at the net section of the plate. In this graphic, *r* is the radius of the hole and *x* is the distance to every node at the net section. The results are compared to the ones obtained with FEM and the experimental [23].

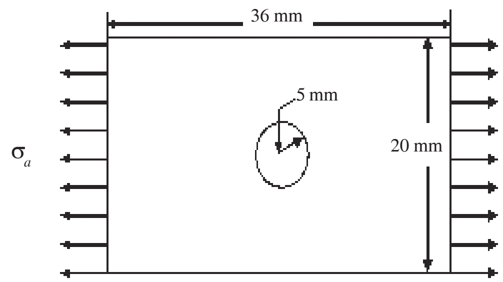

Fig. 3. Geometry and boundary conditions for a perforated plate.

Thick cylinder

The second problem is a thick cylinder which investigates the plane strain expansion subjected to internal pressure. The geometry of a quarter of the cylinder is represented in Figure 4. This part of the cylinder is dicretized into 18 quadratic boundary elements and 24 quadrilateral quadratic internal cells.

The plate has the following material characteristics: $a = 50$mm,

$$E = 120\text{Gpa},\ H' = 0.3,\ H' = 0,\ \sigma_y = 240\text{ Mpa},\ k = \frac{\sigma_y}{\sqrt{3}}\,Mpa.$$

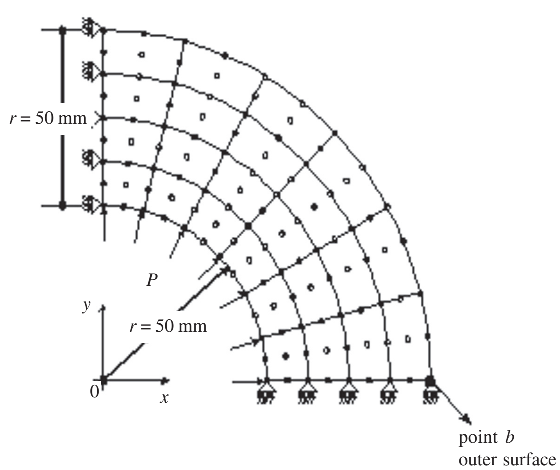

Fig. 4. Geometry, mesh and boundary conditions for a quarter cylinder under pressure.

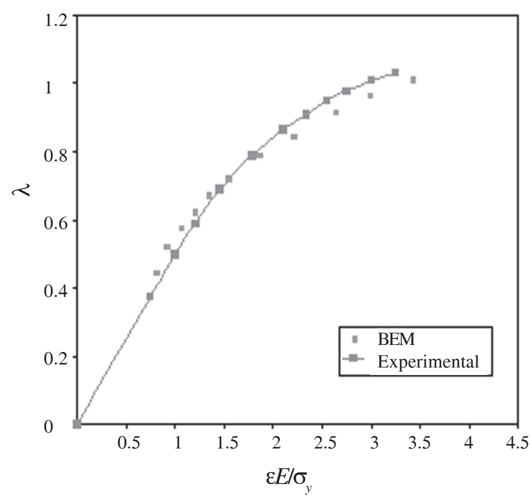

Fig. 5. Development of the deformation with the load factor.

The von Mises yield criterion was applied.

Figure 7 shows the behaviour of the displacement for different loads in a node located at the outer surface of the cylinder. $U(b)$ is the x displacement for every applied load, a is the internal radius and p is the applied load. The outer surface displacements for the plastic front r'=1.6a were calculated and plotted against the load in this Figure 8. After applying internal pressure, the elastic-plastic interface is obtained with

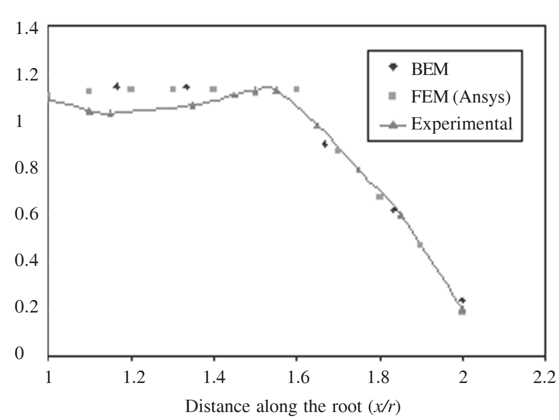

Fig. 6. Stress variation in y direction for the net section of the plate.

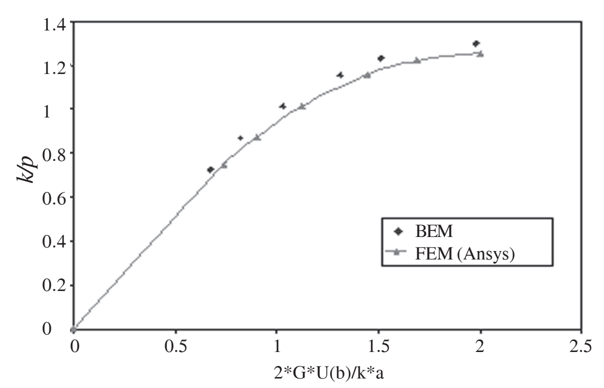

Fig. 7. Outer surface displacements of node *b*.

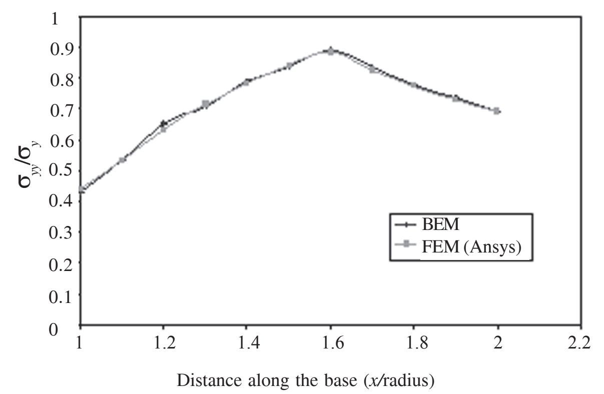

Fig. 8. Stress distribution of a thick cylinder on the base.

BEM which compared with FEM results have good agreement, see figure 8.

7. Conclusions

Two-dimensional elastoplastic analysis has been carried out applying the BEM. An initial strain approach with loading steps was applied in order to execute the non-linar analysis. Successful applications of the BEM are reported here. It was found that the biggest difference in the results compared with the experimental was 3%, but compared with FEM was 1%. The finite element mesh used here was 3 times bigger than that boundary element mesh and this means that it was more computational time in the case of FEM. The BEM compared with the FEM and experimental solutions presents very accurate results which makes this method a very promising tool to analyze nonlinear problems.

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