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Contenido

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EDITORIAL

La revista *Científica*, como su nombre lo indica, es un medio de difusión del conocimiento científico y tecnológico, el cual se caracteriza por publicar artículos científicos originales que contribuyen al conocimiento en diversas áreas de la ciencia y la tecnología, así como artículos donde se presentan una revisión profunda sobre diversos temas de interés actual. Continuando con su característica multidisciplinaria en este número, la revista *Científica* publica artículos relacionados con la ingeniería mecánica, la ingeniería eléctrica y la electrónica.

El primer artículo presenta la validación con datos experimentales obtenidos en el aire al nivel del mar de diversos modelos para el cálculo de campo reducido (E/N) en descargas eléctricas. Aquí se muestra que un parámetro importante que gobierna el comportamiento de la descarga es la relación del campo eléctrico E a la presión p del gas o más exactamente a la densidad de moléculas N del gas.

El segundo trabajo presenta un estudio de la influencia de cuatro de los principales parámetros que intervienen en la eficiencia de separación de un ciclón con entrada tangencial. Aquí se consideran los efectos de algunas propiedades del gas y de las partículas por separar, así como los efectos de algunas dimensiones geométricas de los ciclones.

El siguiente artículo examina tres métodos para diagnóstico paramétrico de turbinas de gas, en el cual el funcionamiento de los diferentes métodos se simula en condiciones idénticas al desarrollo de fallas y errores aleatorios de medición. Los objetivos de esta investigación son afinar los métodos, compararlos y escoger el mejor, con base en criterios probabilísticos para el reconocimiento correcto e incorrecto de las clases de fallas.

En «VSLI Fuzzy Cells» se presenta el desarrollo de celdas básicas para la construcción de funciones de membresía trapezoidales, las cuales se encuentran formadas por un circuito de sustracción de corriente, un multiplicador-divisor y circuitos de forma *S-Z* usando tecnología CMOS de 0.18 μm en modo de corriente.

Finalmente se presenta el uso de técnicas modernas de elementos finitos para obtener eficientemente la respuesta a la frecuencia de máquinas síncronas en reposo, esto responde al continuo interés que existe por evitar pruebas experimentales en máquinas de alta potencia, debido a que implican altos costos y riesgo de daños. El modelo de elementos finitos desarrollado en este trabajo también toma en consideración los circuitos externos conectados a la máquina a través de una solución simultánea de las ecuaciones de los circuitos y del dominio electromagnético.

Así, de esta manera, el presente número publica contribuciones que colaboran e incrementan el conocimiento en campos relativos las ingenierías eléctrica, mecánica y electrónica.

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Numerical Solution for the One-Dimension Heat Equation by a Pseudo-Spectral Discretization Technique

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1. Abstract

The implementation of spectral methods technique for numerically solving partial differential equations has been developed in the past years. Its simple implementation as well as its amazing accuracy for solving partial differential equations makes it the best choice among different numerical techniques. In this work the application of spectral methods for solving the 1-D Heat equation is presented.

Key words: 1-D heat equation, pseudo-spectral methods.

2. Resumen (Solución numérica a la ecuación del calor en una dimensión por medio de métodos seudo-espectrales de discretización)

La implementación de los métodos espectrales para resolver numéricamente ecuaciones diferenciales parciales es una técnica que ha sido desarrollada en los últimos años. Su sencilla implementación y la sorprendente exactitud que presenta en la solución de ecuaciones diferenciales la pone en ventaja en comparación con otras técnicas numéricas. En este trabajo se presenta la aplicación de métodos espectrales para la solución de la ecuación del calor en una dimensión (1-D).

Palabras clave: ecuación del calor en 1-D, métodos seudo-espectrales.

3. Introduction

Much attention has been devoted in the past decades to the development of efficient, accurate and stable numerical schemes for the solution of partial differential equations. Three classes of solution techniques have emerged: the finite difference techniques, the finite element methods, and the spectral techniques.

The last one presents the advantage of high accuracy attained by the resulting discretization for a given number of nodes or, conversely, the saving computational resources for a given accuracy. Specially for problems with a high degree of continuity these methods can lead to very accurate results.

A drawback of spectral methods is that the involved matrices in general are full and hence their inversion can be extremely time-consuming. However, this problem can be solved in several ways [1], for example using relaxation techniques, separation of variables technique, non-stationary iterative techniques, etc.

The spectral methods are known as the Method of Weighted Residuals (MWR). The key elements for developing this technique are the trial functions (approximation functions) and the test functions (weight functions) [2]. The first one is used as the basis functions for a truncated series expansion of the solution. The test functions are used to ensure that the differential equation is closely satisfied by the truncated series expansion. This is achieved by minimizing the residuals or the error in the differential equation that is produced by using the truncated series instead of the exact solution.

The most frequently used trial functions are trigonometric polynomials as Chebyshev polynomias, and Legendre polynomials. In the case of periodic problems, trigonometric interpolants in equispaced points are generally used, while for non-periodic problems; polynomial interpolants in unevenly spaced grid are preferred [3]. For instance, trigonometric polynomials for periodic problems, Legendre polynomials and Chebyshev polynomials for non-periodic problems [4].

The merit of spectral methods is the high accuracy or the spectral accuracy. For example, if we consider numerical solution using finite differences or finite element scheme, the error decreases like $\vartheta(N^{-m})$ for some constant m that depends on the order of approximation and the smoothness of the solution. For a spectral method, convergence at the rate $\vartheta(N^{-m})$ for every m is achieved, provided the solution is infinitely differentiable [4].

The Galerkin, Collocation, and Tau versions are the three most commonly used test functions in the spectral schemes as it is described below.

In the Galerkin approach the test functions are the same as the trial functions. Spectral Tau methods are similar to Galerkin methods, while in the Collocation approach the test functions are translated Dirac delta functions centered at special collocation points [3]. This approach requires that the differential equation must be exactly satisfied at the collocation points.

The most simple of the discretization schemes before mentioned, is the Collocation approach, and is the one presented in this paper. The fundamental principle of spectral collocation methods is given discrete data on a grid to interpolate the data globally, and then evaluate the derivative of the interpolant on the grid [4].

4. Development

4.1 Chebyshev Collocation Method

The Chebyshev polynomials on [–1,1] are defined by [5]:

$$T_k(x) = \cos(k \cos^{-1} x) \qquad (1)$$

Considering the linear heat equation with homogeneous Dirichlet boundary conditions [6] as it is shown in equation (2):

$$\begin{aligned} &\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \\ &u(-1,t) = 0 \\ &u(1,t) = 0 \end{aligned} \qquad (2)$$

And choosing the trial functions as follow [3]:

$$\Phi_k(x)\ T_k(x) \qquad k = 0,1,\ldots,N \qquad (3)$$

Then, the approximate solution has the form:

$$u^N(x,t) = \sum_{k=0}^{N} a_k(t)\phi_k(x) \qquad (4)$$

In the Collocation approach for spectral methods [3], the test functions are the shifted Dirac delta-functions defined in equation (5). In this equation the x_j are the different collocations points in the interval.

$$\Psi_j = \delta(x - x_i) \qquad j = 1, \ldots, N-1 \qquad (5)$$

The standard condition of the weighted residuals condition; reduces to the requirement that the differential equation must be satisfied exactly by the approximate solution at each of the collocation points x_j, as it is presented in equation (6) and (7):

$$\int_{-1}^{1} \left[\frac{\partial u^N}{\partial t} - M(u^N) \right] \Psi_j(x)dx = 0 \qquad j = 1, \ldots, N-1 \qquad (6)$$

$$\frac{\partial u^N}{\partial t} - M(u^N)\big|_{x = x_j} \qquad j = 1, \ldots, N-1 \qquad (7)$$

In the equations (6) and (7), $M(u)$ represents an operator which contains all the spatial derivatives for u.

A convenient choice for the collocation points x_j is:

$$x_j = \cos\left(\pi j / N\right) \qquad (8)$$

note that:

$$\phi_k(x_j) = \cos\left(\pi jk / N\right) \qquad (9)$$

Finally the approximation of the differential equation takes the form [3]:

$$\left.\frac{\partial u^N}{\partial t}\right|_{x=x_j} = \sum_{k=0}^{N} a_k^{(2)}(t)\cos\left(\pi jk/N\right) \qquad (10)$$

4.2 Chebyshev differentiation matrix

Let a function v be defined on the Chebyshev points, and also, let p be the unique polynomial of degree $\leq N$ that satisfies the following condition $p(x_j) = v_j$, $0 \leq j \leq N$.

Setting $w_j = p'(x_j)$, the last statement can be represented as an linear operation, *i.e.* it can be represented by an $(n+1)$x$(n+1)$ matrix, as it is suggested in equation (11).

$$w = D_N v \qquad (11)$$

As example, consider N=2. According to equation (8), the collocations points are x_j=[1,0,–1].

The polynomial $p(x)$ that satisfies the before statement can be computed using the Lagrange interpolation [8]. The result is presented in equation (12) [4].

$$p(x) = \frac{1}{2}x(1+x)v_0 + (1+x)(1-x)v_1 + \frac{1}{2}x(x-1)v_2 \qquad (12)$$

And its derivative is:

$$p'(x) = \left(x+\frac{1}{2}\right)v_0 - 2xv_1 + \left(x-\frac{1}{2}\right)v_2 \qquad (13)$$

The differentiation matrix D_2 is a 3x3 matrix whose rows are obtained by substituting the values of the collocations points for the expression above as presented in equation (14) [4].

$$D_2 = \begin{bmatrix} 3/2 & -2 & 1/2 \\ 1/2 & 0 & -3/2 \\ -1/2 & 2 & -3/2 \end{bmatrix} \qquad (14)$$

A general formulation for the entries of D_N for any integer N is shown in Figure 1 [3]. Note that, the second derivative matrix $D_N^{\ 2}$ must be the square of D_N [3].

$$(D_N)_{ij} = \begin{cases} \dfrac{c_i(-1)^{i+j}}{c_j(x_i - x_j)} & i = j \\[2ex] \dfrac{-x_j}{2(1-x_j^2)} & 1 \leq i = j \leq 1, \ldots, N-1 \\[2ex] \dfrac{2N^2+1}{6} & i = j = 1 \\[2ex] -\dfrac{2N^2+1}{6} & i = j = N \end{cases}$$

$$c_i = \begin{cases} 2 & i = 1 \text{ or } N \\ 1 & \text{otherwise} \end{cases}$$

Fig. 1. Entries for a Chebyshev differentiation matrix.

4.3 1-D Parabolic heat equation

Let's consider as an application example the one-dimensional heat equation. The general equation for this problem is presented in equation (15).

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \qquad (15)$$

The boundary and initial conditions are:

B.C.

$$u(-L,t) = u(L,t) = 0 \qquad (16)$$

I.C.

$$u(x,0) = a_0 + \sum_{n=1}^{N}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

This equation is an example of parabolic PDE with constant coefficients. Where α is the thermal diffusivity.

If we use the technique of separation of variables to find the analytical solution, and substitute the values of the boundary and initial condition the general solution has the following form [7]:

$$u(x,t) = a_0 + \sum_{n=1}^{N} e^{-(n\pi/L)^2 \alpha t} \left[\begin{matrix} a_n \cos\left(\frac{n\pi x}{L}\right) + \\ b_n \sin\left(\frac{n\pi x}{L}\right) \end{matrix} \right] \qquad (17)$$

As can be seen, the numerical solution of the 1-D Heat equation requires discretization in space and time. Spatial discretization relates directly to the solution of the most important small-scale features of the heat transfer as well as the size of the problem. On the other hand, the temporal discretization is related to the unsteady phenomena, but also dictates directly the form of the semi discrete equations to be solved.

7. Application example

For the numerical application example, considerer the parabolic heat equation expressed in equation (15), with initial and boundary conditions as follow:

B.C.

$$u(-1,t) = u(1,t) = 0$$

I.C. (18)

$$u(x,0) = \sin \pi x$$

Also, by simplicity consider L=1 and α=1. Applying the general solution equation (17), we get the analytical solution as:

$$u(x,t) = \mathrm{e}^{-\pi^2 t} \sin \pi x \qquad (19)$$

Resuming what was said about the main objective of the spectral collocations methods, we seek for a solution for the PDE of the form:

$$u_N(x,j) = \sum_{j=0}^{N} u(x_j,t) l_j(x) \qquad (20)$$

Where $l_j(x)$ are the interpolating Chebyshev polynomial based on the Chebyshev-Gauss-Lobbato (CGL) quadrature points as equation (9). It is required that the residual defined by equation (21), vanishes at all the interior points to obtain the equation (22).

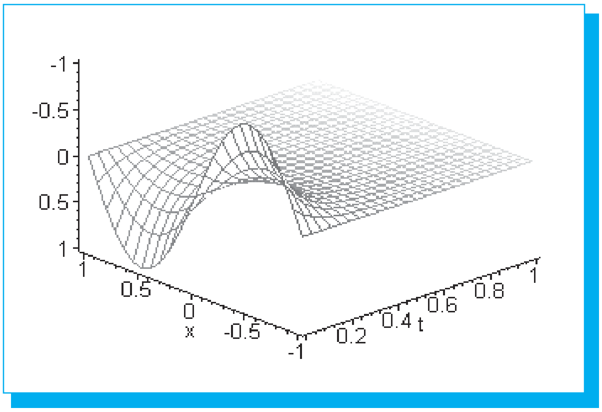

Fig. 2. Analytical solution of the parabolic 1-D heat equation, 0 < t < 1.

$$R_N(x,t) = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \qquad (21)$$

$$\frac{\partial u(x,t)_i}{\partial t} - D^2_{ij} u(x,t)_i = 0 \qquad (22)$$

Where D^2_{ij} represents the second derivative matrix. A glance to the equation before could show that:

$$\frac{\partial u(x,t)_i}{\partial t} = D^2_{ij} u(x,t)_i \qquad (23)$$

or

$$\int \frac{\partial u(x,t)_i}{\partial t} = \int_{t=t_1}^{t=t_2} D^2_{ij} u(x,t)_i \qquad (24)$$

integrating

$$u(x,t)_i = \int_{t=t_1}^{t=t_2} \left[D^2_{ij} u(x,t)_i \right] dt$$

This mean that the integration from $t = t_1$ to $t = t_2$ of the multiplication between the second derivative Chebyshev matrix and the polynomial approximation of the function gives the solution at each collocation point warranting the residual (error) equal to zero. However, the time integration must be made using a very small time steps in order to minimize the time error [4] or instability of the problem. High order methods as a four order Runge-Kutta Method [8] are chosen to handle this situation.

To solve this problem, a double precision computer program was written. The problem was solved for the conditions

Table 1. Maximum error for different order expansion. (Chebyshev-Gauss-Lobatto Collocation points)

N	*Maximum error*
8	2.40E-8
12	3.99E-13
16	4.65E-13
24	4.78E-13
32	4.70E-13
48	4.80E-13
64	4.82E-13
128	4.79E-13

presented in equation (18) and $0 \leq t \leq 1$ as time interval. The analytical solution is plotted in Figure 2.

The polynomial approximation was taken as the solution of the initial condition at each collocation point. Remember that this initial condition must be satisfied in order to get the analytical solution.

Once, the second derivative matrix and the polynomial approximation are found, the integration (see equation 24) of the product of the second derivative matrix and the vector approximation (polynomial) at each collocation point is made using a fourth–order Runge-Kutta time-integration scheme. The time integration is made for a

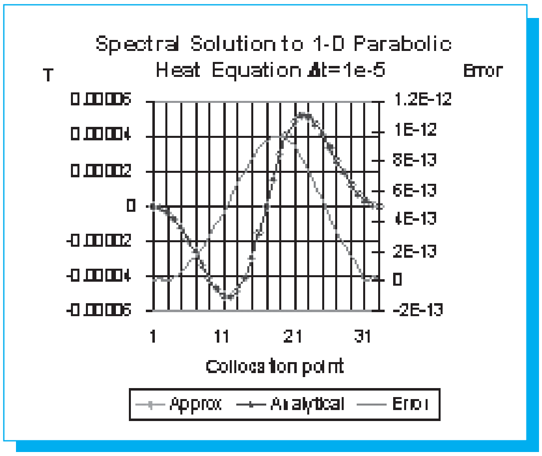

Fig. 3. Analytical, numerical and error for t = 1 at Chebyshev collocation points.

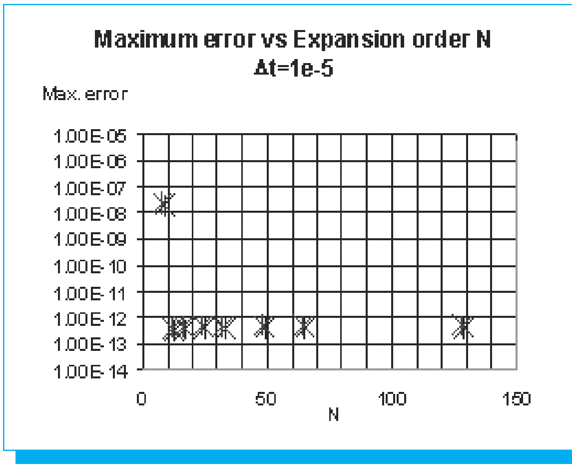

Fig. 4. Maximum error *vs.* expansion order for numerical solution.

time step ($\Delta t = 1x10^{-5}$) in order to avoid run-off errors, as well as to ensure a good approximation.

The program was tested for several order expansion of grid points (N). At first instance, a comparison between the analytic solution and the numerical approximation by pseudo-spectral discretization technique is plotted in Figure 3 for an order expansion $N = 32$. In this figure, it can be appreciated that the numerical approximation had spectral accuracy because the maximum error is on machine precision's magnitude order.

The results of the maximum error for each test is presented in Table 1, and plotted in Figure 4. As can be seen in Figure 4, maximum error decays fast as N increases, in this condition we can said that spectral convergence is obtained for $N \geq 12$.

5. Conclusions

In this work the implementation of spectral methods for solving the 1-D parabolic heat equation was presented. The results showed a high accuracy obtained when this method is applied.

The maximum error for the numerical approximation decays fast even though the expansion order is small. This means that spectral convergence is obtained even at $N \geq 12$, for this example.

For each collocation point the magnitude order of the error was approx. $1x10^{-13}$, which is the principal reason to choose this technique for solving partial differential equations. The same problem was solved using finite differences at the error had a magnitude order of $1x10^{-6}$ with $N = 250$. (This result was obtained after several tests. However the main objective of this job is referred to Spectral Methods, not Finite Differences.)

Nomenclature

CGL Chebyshev-Gauss-Lobbato collocation points
D derivative matrix
l interpolating Chebyshev polynomial at CGL points
M first derivative operator
p polynomial function
R residual
t time
u dependent variable
w first derivative for the polynomial function
x spatial coordinate

Greek symbols

α diffusion coefficient
δ Delta Dirac function
υ function defined at Chebyshev points
ψ test function
ϕ trial function

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