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Mathematical modelling of the mass-spring-damper system - A fractional calculus approach

Modelado matemático del sistema masa-resorte-amortiguador - Enfoque en cálculo fraccionario

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ABSTRACT

In this paper the fractional differential equation for the mass-spring-damper system in terms of the fractional time derivatives of the Caputo type is considered. In order to be consistent with the physical equation, a new parameter is introduced. This parameter characterizes the existence of fractional components in the system. A relation between the fractional order time derivative and the new parameter is found. Different particular cases are analyzed.

RESUMEN

En este trabajo se presenta la ecuación diferencial fraccionaria del sistema masa-resorte-amortiguador en terminos de la derivada fraccionaria de tipo Caputo. Con la finalidad de ser consistentes con la ecuación física, se introduce un nuevo parámetro. Este parámetro caracteriza la existencia de componentes fraccionarias en el sistema. Se encuentra la relación entre el orden de la derivada fraccionaria y el nuevo parámetro. Diferentes casos particulares son analizados.

INTRODUCTION

Fractional calculus (FC), involving derivatives and integrals of non-integer order, is the natural generalization of the classical calculus [1-4]. Many physical phenomena have “intrinsic" fractional order description and, so, FC is necessary in order to explain them. In many applications, FC provides more accurate models of the physical systems than ordinary calculus do. Since its success in description of anomalous diffusion [5], non-integer order calculus, both in one and multidimensional space, has become an important tool in many areas of Physics, Mechanics, Chemistry, Engineering, Finances and Bioengineering [6-9]. Fundamental physical considerations in favor of the use of models based on derivatives of non-integer order are given in [10-12]. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantage of FC, in comparison with the classical integer-order models -in which such effects are in fact neglected.

In [13] are discussed the fractional oscillator equation involving fractional time derivatives of the Riemann-Liouville type [14], considered the linearly damped oscillator equation with the damping term generalized to a Caputo fractional derivative. A solution is found analytically and a comparison with the ordinary linearly damped oscillator is made. Despite introducing the fractional time derivatives, the cases mentioned above seem to be justified; there is no clear understanding of the basic reason for fractional derivation in physics. Therefore, it is interesting to analyze a simple physical system and try to understand their fully behavior given by the fractional differential equation.

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The present work is interested in the study of a simple mechanical system consisting of a mass, a damped coefficient and a spring in the framework of the fractional derivative. The position to follow: in order to change the time derivative operator d/dt by a fractional operator d^γ/dt^γ (γ represents the order of the derivative), it is necessary to introduce an additional parameter σ, which must have dimension of seconds (in the case of the time derivative) to be consistent with the dimension of the ordinary derivative. The σ parameter characterizes the presence of fractional time components in the system.

METHODS

An oscillating system, in general, is a mechanical system consisting of three kinds of elements: a mass (m) measured in kg, a damped coefficient β measured in Ns/m and a spring constant k measured in N/m. The change with respect to time of the movement $x(t)$ is described by the second order homogeneous differential equation

$$m\frac{d^2x(t)}{dt^2}+\beta\frac{dx(t)}{dt}+kx(t)=0. \tag{1}$$

In a real oscillating system, the damped coefficient β is different from zero and the free mechanical oscillations become extinct due to the Joule effect. To compensate for the damping mechanical oscillations, a source $v(t)$ should be include in the system. Therefore, the differential equation that governs the behavior of the system (mass-spring-damper) with source has the form

$$m\frac{d^2x(t)}{dt^2}+\beta\frac{dx(t)}{dt}+kx(t)=v(t). \tag{2}$$

The term $kx(t)$ is very important because lack of it in equations (1) and (2) imply that it has no oscillating system. For the equations (1) and (2), it will be considered the following special cases:

In the absence of the damped coefficient in the system, *i.e.* $\beta = 0$, it is had from (1) and (2)

$$m\frac{d^2x(t)}{dt^2}+kx(t)=0, \tag{3}$$

and

$$m\frac{d^2x(t)}{dt^2}+kx(t)=v(t). \tag{4}$$

In the case when $m = 0$, *i.e.* in the absence of mass in the system, they are had from equations (1) and (2)

$$\beta\frac{dx(t)}{dt}+kx(t)=0, \tag{5}$$

and

$$\beta\frac{dx(t)}{dt}+kx(t)=v(t). \tag{6}$$

The solutions of the equations (1) and (2) are well known.

It is important to note that fractional differential equations corresponding to oscillating systems have been studied before, replacing the integer time derivative by fractional one on a purely mathematical or heuristics basis [14].

The idea is to write the equations (1) and (2) in terms of a fractional time derivative operator. It is proposed to change the ordinary time derivative operator by fractional one, in the following way

$$\frac{d}{dt}\rightarrow\frac{d^\gamma}{dt^\gamma}, \qquad 0<\gamma\leq 1, \tag{7}$$

where γ is an arbitrary parameter representing the order of the fractional time derivative operator, and in the case $\gamma = 1$ becomes ordinary time derivative operator. Since the time derivative operator has dimensions of inverse seconds s^{-1}, then from (7) it is seen that the expression

$$\left[\frac{d^\gamma}{dt^\gamma}\right]=\frac{1}{s^\gamma}, \tag{8}$$

it is not a time derivative operator in the usual sense, because its dimensionality is $s^{-\gamma}$. In order to be consistent with time dimensionality, it is introduced the parameter σ in the following way

$$\left[\frac{1}{\sigma^{1-\gamma}}\frac{d^\gamma}{dt^\gamma}\right]=\frac{1}{s}, \qquad 0<\gamma\leq 1 \tag{9}$$

such that, when $\gamma = 1$, the expression (9) becomes ordinary derivative operator d/dt . This dimensionality equation is satisfied, only if the parameter σ has dimension of seconds, $[\sigma] = s$. Therefore, it is possible to always change ordinary time derivative operator by the fractional one in the following general form

$$\frac{d}{dt}\rightarrow\frac{1}{\sigma^{1-\gamma}}\frac{d^\gamma}{dt^\gamma}, \qquad n-1<\gamma\leq n, \tag{10}$$

where n is integer. These two expressions represent time derivatives, as the dimensions are inverse seconds s^{-1}. The parameter $\sigma^{1-\gamma}$ can be called a fractional time parameter representing the fractional time components in the system; its dimensionality is $s^{1-\gamma}$. This non-local time is called in the literature the *cosmic time* [15]. Another physical and geometrical interpretation of the fractional operators is given in [16].

Hopefully, that fractional time derivative operator defined in (10) will be very useful in the construction of fractional differential equations for the physical and engineering problem.

On the other hand, Caputo's fractional derivative of a function $f(t)$ is defined as an integral transform of ordinary derivative to fractional one [4]:

$$ {}_0^C D_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(\eta)}{(t-\eta)^{\gamma-n+1}} d\eta, \quad (11)$$

where C indicates Caputo derivative, evaluated in 0 and order γ , $n = 1,2,... \in N$ and $n - 1 < \gamma \leq n$. It is considered the case $n = 1$, *i.e.* in the integrand there is only first derivative. In this case $0 < \gamma \leq 1$ is the interval of the fractional derivative.

For example, in the case $f(t) = t^k$, where k is arbitrary number and $0 < \gamma \leq 1$, it is had the following expression for the fractional derivative operation (which will be useful later for solving the wave equation with fractional time derivative ${}_0^C D_t^\gamma \equiv d^\gamma / dt^\gamma$):

$$ {}_0^C D_t^\gamma t^k = \frac{k\Gamma(k)}{\Gamma(k+1-\gamma)} t^{k-\gamma}, \quad (0 < \gamma \leq 1), \quad (12)$$

where $\Gamma(k)$ and $\Gamma(k + 1- \gamma)$ are the Gamma functions. If $\gamma = 1$, the expression (12) yields to ordinary derivative

$$ {}_0^C D_t^1 t^k = \frac{dt^k}{dt} = kt^{k-1}. \quad (13)$$

RESULTS

Now, it is possible to write the ordinary differential equations (1-6) in the fractional form. Using the expression (10), the fractional differential equations corresponding to equations (1) and (2) are given by

$$ \frac{m}{\sigma^{2(1-\gamma)}} \frac{d^{2\gamma}x(t)}{dt^{2\gamma}} + \frac{\beta}{\sigma^{1-\gamma}} \frac{d^\gamma x(t)}{dt^\gamma} + kx(t) = 0, \quad 0 < \gamma \leq 1, \quad (14)$$

and

$$ \frac{m}{\sigma^{2(1-\gamma)}} \frac{d^{2\gamma}x(t)}{dt^{2\gamma}} + \frac{\beta}{\sigma^{1-\gamma}} \frac{d^\gamma x(t)}{dt^\gamma} + kx(t) = \upsilon(t), \quad 0 < \gamma \leq 1, \quad (15)$$

from these equations can be deduced the particular cases, when $\beta = 0$

$$ \frac{m}{\sigma^{2(1-\gamma)}} \frac{d^{2\gamma}x(t)}{dt^{2\gamma}} + kx(t) = 0, \quad 0 < \gamma \leq 1, \quad (16)$$

$$ \frac{m}{\sigma^{2(1-\gamma)}} \frac{d^{2\gamma}x(t)}{dt^{2\gamma}} + kx(t) = \upsilon(\mathrm{t}), \quad 0 < \gamma \leq 1, \quad (17)$$

and $m = 0$

$$ \frac{\beta}{\sigma^{1-\gamma}} \frac{d^\gamma x(t)}{dt^\gamma} + kx(t) = 0, \quad 0 < \gamma \leq 1, \quad (18)$$

$$ \frac{\beta}{\sigma^{1-\gamma}} \frac{d^\gamma x(t)}{dt^\gamma} + kx(t) = \upsilon(t), \quad 0 < \gamma \leq 1. \quad (19)$$

From the equation (16), it is identified

$$ \omega^2 = \frac{k\sigma^{2(1-\gamma)}}{m} = \omega_0^2 \sigma^{2(1-\gamma)}, \quad (20)$$

as the fractional angular frequency, where $\omega_0^2 = k/m$ is the fundamental frequency of the non-fractional system (*i.e*, when $\gamma = 1$).

On the other hand, from equation (14), when $k = 0$, it is defined the fractional relaxation time as

$$ \tau_\gamma = \frac{m}{\beta\sigma^{1-\gamma}}. \quad (21)$$

This fractional relaxation time arises in the experimental studies of the complex systems [12]. When $\gamma = 1$ from this expression, it is had the well known relaxation time $\tau = m/\beta$.

Solutions of the equations (14) and (15) may be obtained using the Laplace transform. Solutions of the homogeneous fractional differential equations (16) and (18) will be found in terms of the Mittag-Leffler function [17].

Then, using definition (11), the homogeneous solution of the equation (16) is given by

$$ x_h(t) = x_0 E_{2\gamma}\left\{-\frac{\sigma^{2(1-\gamma)}k}{m} t^{2\gamma}\right\}, \quad (22)$$

where

$$ E_{2\gamma}\left\{-\frac{\sigma^{2(1-\gamma)}k}{m} t^{2\gamma}\right\} = \sum_{n=0}^{\infty} \frac{\left(-\frac{\sigma^{2(1-\gamma)}k}{m} t^{2\gamma}\right)^n}{\Gamma(2\gamma n+1)}, \quad (23)$$

is the Mittag-Leffler function and $x(0) = x_0$.

First case: $\gamma = 1$, the expression (23) becomes hyperbolic cosine

$$ E_2\left\{-\frac{k}{m}t^2\right\} = \mathrm{ch}\left(\sqrt{-\frac{k}{m}t^2}\right) = \mathrm{ch}\left(\frac{i\sqrt{k}}{\sqrt{m}}t\right) = \cos\left(\frac{t\sqrt{k}}{\sqrt{m}}\right). \quad (24)$$

Then in this case, it is had periodic function given by

$$ \tilde{x}_h(t) = x_0 \cos(\omega_0 t), \quad (25)$$

with angular frequency, $\omega_0 = \sqrt{k/m}$. The expression (25) is the well known solution of the integer differential equation (3).

Second case: $\gamma = 1/2$, from (22), it is given

$$ \hat{x}_h(t) = x_0 E_1\left\{-\frac{\sigma k}{m}t\right\} = x_0 e^{-\frac{\sigma k}{m}t}. \quad (26)$$

Note that the parameter γ -which characterizes the fractional order time derivative- can be related to the σ parameter -which characterizes the existence, in the system, of fractional excitations [18]-[19]. For example, for the system described by the fractional equation (16), it is possible to write the relation

$$\gamma = \frac{\sigma}{\sqrt{m / k}}, \qquad 0 < \sigma \leq \sqrt{\frac{m}{k}}. \tag{27}$$

Then, the magnitude $\delta = 1 - \gamma$ characterizes the existence of fractional structures in the system. It is easy to see that when $\gamma = 1$, then $\sigma = \sqrt{m/k} = \omega_0$ and therefore $\delta = 0$, that means that in the system there are not fractal structure. However, in the case $0 < \gamma < \sqrt{m/k}$, δ grows and tends to unity, because in the system are increasingly fractal excitations. Taking account the expression (27), the solution (22) of the equation (16) can be rewritten through γ, as follows

$$x_h(t) = x_0 E_{2\gamma}\left\{-\gamma^{2(1-\gamma)}\left(\frac{t}{\sqrt{m / k}}\right)^{2\gamma}\right\}. \tag{28}$$

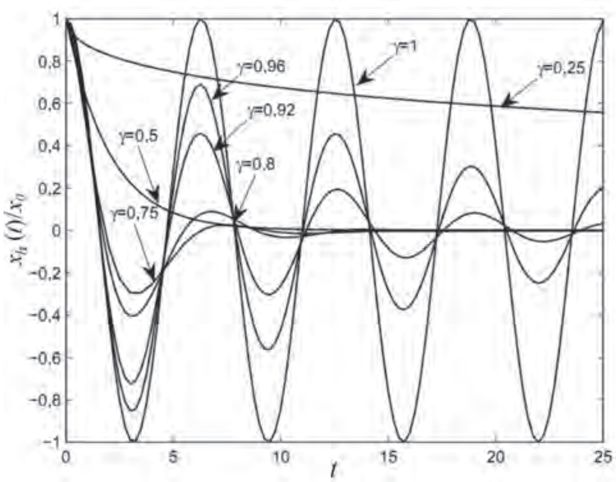

Figure 1. Mass-Spring System, $\gamma = 1$, $\gamma = 0{,}96$, $\gamma = 0{,}92$, $\gamma = 0{,}8$, $\gamma = 0{,}75$, $\gamma = 0{,}50$ and $\gamma = 0{,}25$.

Figure 1, shows the solution of (28) for different values of γ.

Respect to the figure 1, the displacement of the fractional oscillator is essentially described by the Mittag-Leffler function for the considered initial conditions. The function is

$$E_{2\gamma}\left\{-\gamma^{2(1-\gamma)}\left(\frac{t}{\sqrt{m/k}}\right)^{2\gamma}\right\}$$

It is showed by numerical calculations that the displacement of the fractional oscillator varies as a function of time and how this time variation depends on the parameter γ. Also, it is proved that, if γ is less than 1, the displacement shows the behavior of a damped harmonic oscillator. As a result, in consistent with the case of simple harmonic oscillator, the total energy of simple fractional oscillator will not be a constant. The damping of fractional oscillator is intrinsic to the equation of motion, and not introduced by additional forces as in the case of a damped harmonic oscillator. A simple fractional oscillator behaves like a damped harmonic oscillator.

Solution of the equation (18) is given by

$$\tilde{x}_\gamma(t) = \tilde{x}_0 E_\gamma\left\{-\frac{\sigma^{1-\gamma}k}{\beta}t^\gamma\right\} = \tilde{x}_0 \sum_{n=0}^{\infty} \frac{\left(-\frac{\sigma^{1-\gamma}k}{\beta}t^\gamma\right)^n}{\Gamma(\gamma n+1)}, \tag{29}$$

which in the case $\gamma = 1$, reduces to

$$\tilde{x}_0(t) = \tilde{x}_0 e^{-\frac{kt}{\beta}}, \tag{30}$$

which is the well known solution of the integer differential equation (5). In this case the relation between γ and σ is given by

$$\gamma = \frac{\sigma k}{\beta}, \qquad 0 < \sigma \leq \frac{k}{\beta}. \tag{31}$$

The solution (29) of the fractional equation (18), taking account the relation (31), may be written as

$$x(t) = x_0 E_\gamma\left\{-\gamma^{(1-\gamma)}\left(\frac{kt}{\beta}\right)^\gamma\right\}. \tag{32}$$

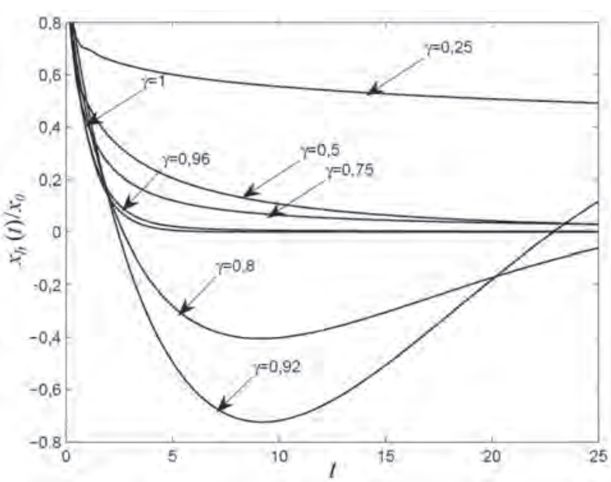

Figure 2. Damper-Spring System, $\gamma = 1$, $\gamma = 0{,}96$, $\gamma = 0{,}92$, $\gamma = 0{,}8$, $\gamma = 0{,}75$, $\gamma = 0{,}50$ and $\gamma = 0{,}25$.

Figure 2, shows the solution of (32) for different values of γ.

The equations (17) and (19) are fractional linear non-homogeneous differential equations, then, the general solution is given by the sum of the homogeneous (16) and (18) solutions and particular solutions. These solutions may be obtained applying direct and inverse Laplace transform. As result, the solution for the fractional non homogenous equation (17) is given by

$$x_\gamma(t) = x_0 E_{2\gamma}\left\{-\gamma^{1-\gamma}\left(\frac{t}{\sqrt{m/k}}\right)^{2\gamma}\right\} + \int^t \upsilon(\tau) E_{2\gamma}\left\{-\gamma^{1-\gamma}\left(\frac{t-\tau}{\sqrt{m/k}}\right)^{2\gamma}\right\} d\tau. \quad (33)$$

where it has been used the expression (27).

Taking account the expression (31), the solution of the fractional non homogeneous equation (19) has the form

$$\tilde{x}_\gamma(t) = \tilde{x}_0 E_\gamma\left\{-\gamma^{1-\gamma}\left(\frac{t\beta}{k}\right)^{\gamma}\right\} + \int^t \upsilon(\tau) E_\gamma\left\{-\gamma^{1-\gamma}\left(\frac{\beta(t-\tau)}{k}\right)^{\gamma}\right\} d\tau. \quad (34)$$

It easy to see that, in the case $\gamma = 1$, the solution (34), becomes

$$\tilde{x}_1(t) = \tilde{x}_0 e^{-\beta t/k} + \int^t \upsilon(\tau) e^{-\frac{\beta(t-\tau)}{k}} d\tau = \tilde{x}_0 e^{-\beta t/k} + e^{-\beta t/k}\int^t \upsilon(\tau) e^{\beta\tau/k} d\tau, \quad (35)$$

which represents the general solution of the ordinary integer equation (6).

Now it is possible to pass to the general solutions of equations (14) and (15) for a mechanical system having all mechanical elements; mass m, damped coefficient β and spring k. The solution of the equation (14) has the form

$$x_\gamma(t) = x_0 E_\gamma\left\{-\frac{\beta\sigma^{1-\gamma}}{2m} t^\gamma\right\} \times E_{2\gamma}\left\{-\left[\frac{k}{m} - \frac{\beta^2}{4m^2}\right]\sigma^{2(1-\gamma)} t^{2\gamma}\right\}. \quad (36)$$

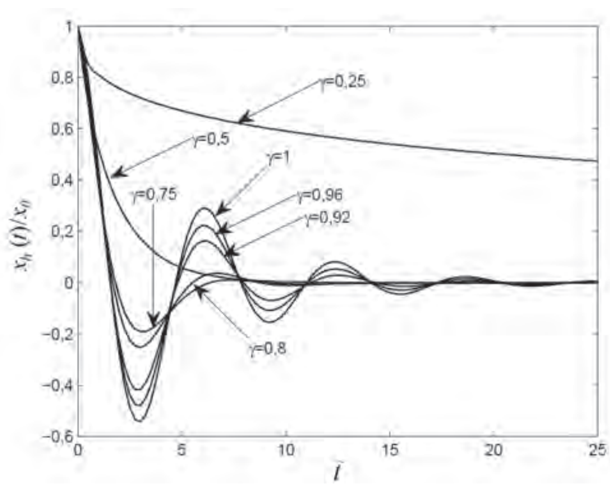

Figure 3. Solution of (36) for different values of γ, $\gamma = 1$, $\gamma = 0{,}96$, $\gamma = 0{,}92$, $\gamma = 0{,}8$, $\gamma = 0{,}75$, $\gamma = 0{,}50$ and $\gamma = 0{,}25$.

Figure 3, shows the solution of (36) for different values of γ.

From equation (36), it is clear that in the case $\gamma = 1$, it is had

$$x_1(t) = x_0 e^{-\beta t/2m}\cos\left(\sqrt{\frac{k}{m} - \frac{\beta^2}{4m^2}}\, t\right). \quad (37)$$

In this case, they are had damping vibrations due to the presence of the damped coefficient $\beta \neq 0$. The solution (37) is performed in $R < 2\sqrt{km}$ and has two constants of integrations as it should be for second ordinary differential equations (1) ($\beta/2m = \chi$ is the coefficient of damping of vibrations). In this case, the parameter γ and σ have the following relation

$$\gamma = \left(\frac{k}{m} - \frac{\beta^2}{4m^2}\right)^{1/2}\sigma, \qquad 0 < \sigma \le \frac{1}{(k/m - \beta^2/4m^2)^{1/2}}. \quad (38)$$

Then, the solution takes the form (36)

$$x_\gamma(t) = x_0 E_\gamma\left\{-\frac{\beta}{2m\sqrt{k/m - \beta^2/4m^2}}\gamma^{(1-\gamma)}\tilde{t}^{\gamma}\right\} \times E_{2\gamma}\left\{-\gamma^{2(1-\gamma)}\tilde{t}^{2\gamma}\right\}, \quad (39)$$

where

$$\tilde{t} = \left(\frac{k}{m} - \frac{\beta^2}{4m^2}\right)^{1/2} t \quad (40)$$

Because of condition $\beta < 2\sqrt{km}$, it is possible to choose as an example

$$\frac{\beta}{2m\sqrt{k/m - \beta^2/4m^2}} = 2, \quad 0 \le \frac{\beta}{2m\sqrt{k/m - \beta^2/4m^2}} < \infty. \quad (41)$$

Finally, the solution (36) takes the following form

$$x_\gamma(t) = E_\gamma\left\{-2\gamma^{(1-\gamma)}\tilde{t}^{\gamma}\right\} \times E_{2\gamma}\left\{-\gamma^{2(1-\gamma)}\tilde{t}^{2\gamma}\right\}. \quad (42)$$

For the case $\beta < 2\sqrt{km}$, the solution of the equation (14) has the form

$$\tilde{x}_\gamma(t) = x_0 E_\gamma\left\{-\frac{\beta\sigma^{(1-\gamma)}}{2m} t^\gamma\right\} \times E_\gamma\left\{-\left[\frac{\beta^2}{4m^2} - \frac{k}{m}\right]^{1/2}\sigma^{(1-\gamma)} t^\gamma\right\}. \quad (43)$$

In the case when $\gamma = 1$, it is had

$$\tilde{x}_1(t) = x_0 e^{-\beta/2m(1+\sqrt{1-4m/\beta^2 k})t}, \quad (44)$$

where $x(0) = x_0$ is the movement in the spring in $t = 0$. The solution (44) corresponds to the equation (1) and characterize the change of movement $x(t)$ on the spring and has aperiodic character.

Taking the following relations

$$\gamma = (\beta^2/4m^2 - k/m)^{1/2}\sigma, \quad 0 < \sigma \le \frac{1}{(\beta^2/4m^2 - k/m)^{1/2}}, \quad (45)$$

the solution (43) takes the form

$$\tilde{x}_\gamma(t) = x_0 E_\gamma\left\{-\frac{\beta}{2m\sqrt{\frac{\beta^2}{4m^2}-\frac{k}{m}}}\gamma^{(1-\gamma)}\tilde{t}^{\gamma}\right\} \times E_\gamma\left\{-\gamma^{(1-\gamma)}\tilde{t}^{\gamma}\right\}, \quad (46)$$

where

$$\tilde{t} = (\beta^2/4m^2 - k/m)^{1/2}t\,. \quad (47)$$

If the condition $\beta < 2\sqrt{km}$ is fullfilled, it is given the following region range of values

$$1 < \frac{\beta}{2m\sqrt{\beta^2/4m^2 - k/m}} < \infty, \quad (48)$$

it is possible to choose

$$\frac{\beta}{2m\sqrt{\beta^2/4m^2 - k/m}} = 2\,, \quad (49)$$

then, the solution (46) can be written as

$$\tilde{x}_\gamma(\tilde{t}) = x_0 E_\gamma\left\{-2\gamma^{(1-\gamma)}\tilde{t}^{\gamma}\right\}\times E_\gamma\left\{-\gamma^{(1-\gamma)}\tilde{t}^{\gamma}\right\}. \quad (50)$$

Figure 4, shows the solution of (50) for different values of γ.

Figures 3 and 4 show the complete solution of the system mass-spring-damper. Figure 3 and 4 show that the energy is conserved in the oscillator -is conservative when γ = 1-, while the fractional oscillator (1/2 < γ < 1) show a dissipative nature.

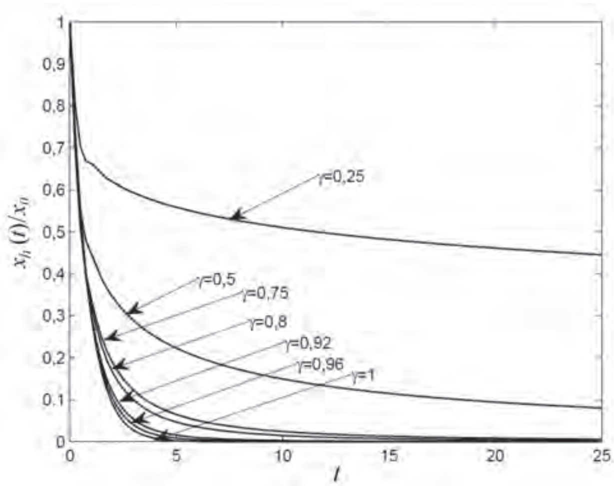

Figure 4. Solution of (50) for different values of γ, γ = 1, γ = 0,96, γ = 0,92, γ = 0,8, γ = 0,75, γ = 0,50 and γ = 0,25.

CONCLUSSION

Fractional calculus is a very useful tool in describing the evolution of systems with memory, which typically are dissipative and to complex systems. In recent decades, it has attracted interest of researches in several areas of science. Specially, in the field of physics applications of fractional calculus have gained considerable popularity [20-21]. In spite of these various applications, there are some important challenges. For example, physical interpretation for the fractional derivative is not completely clarified yet [15].

It has been presented a new fractional differential equation for the oscillating systems. The proposed equation gives a new universal behavior for the oscillating systems characterizing the existence of the fractal structures on the system. It was also found out that there is a relation between γ and σ depending on the system in studies.

In all simulations, the computational time is around 42,43 s using an Intel Core 2 Duo, 1,8 GHz, 2,99 GB RAM.

With the approach presented here, it will be possible to have a better study of the transient effects in the mechanical systems.

The discussion of the solutions (28) and (32), the general case of the equation (14) with respect to the parameter γ, the classification of fractal systems depending on the magnitude of δ and a complete analysis of the solutions (33) and (34) for different value of will be made in a future paper.

It is hoped that this way of dealing with fractional differential equations may help to understand the behavior of the fractional order systems better.

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